**Math 256B. Solutions to Homework 1**

1. (10 points) Hartshorne II Ex. 1.20: *Subsheaf with Supports.* Let $Z$ be a closed subset of $X$, and let $\mathscr{F}$ be a sheaf on $X$. We define $\Gamma_Z(X,\mathscr{F})$ to be the subgroup of $\Gamma(X,\mathscr{F})$ consisting of all sections whose support (Ex. 1.14) is contained in $Z$.

    (a). Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V,\mathscr{F}\big|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in $Z$, and is denoted $\mathscr{H}^0_Z(\mathscr{F})$.

    (b). Let $U = X - Z$, and let $j\colon U \to X$ be the inclusion. Show that there is an exact sequence of sheaves on $X$

$$0 \longrightarrow \mathscr{H}^0_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*\big(\mathscr{F}\big|_U\big)\ .$$

    Furthermore if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*\big(\mathscr{F}\big|_U\big)$ is surjective.

**(a)**. The identity axiom ((3) on page 61) is immediate from the fact that the presheaf in question is by definition a subpresheaf of the sheaf $\mathscr{F}$.

For the gluing axiom (4), let $U$ be an open subset of $X$, let $\{V_i\}$ be an open covering of $U$, and for each $i$ let $s_i \in \Gamma_{Z\cap V_i}(V_i,\mathscr{F}\big|_{V_i})$ be sections that agree on $V_i \cap V_j$ for all $i,j$. They are also sections of $\mathscr{F}$, so they can be glued to give a section $s \in \mathscr{F}(U)$ such that $s\big|_{V_i} = s_i$ for all $i$. By definition $s_P = 0$ for all $P \in U \setminus Z$, so in fact $s$ is in $\Gamma_{Z\cap U}(U,\mathscr{F}\big|_U)$. Thus $\mathscr{H}^0_Z(\mathscr{F})$ is a sheaf.

**(b)**. By construction, $\mathscr{H}^0_Z(\mathscr{F})$ is a subsheaf of $\mathscr{F}$, so we let the map $\mathscr{H}^0_Z(\mathscr{F}) \longrightarrow \mathscr{F}$ be the inclusion map (which is injective).

To construct the map $\beta\colon \mathscr{F} \to j_*(\mathscr{F}\big|_U)$, first note that $j_*(\mathscr{F}\big|_U)$ is the sheaf $V \mapsto \mathscr{F}(U\cap V)$. Then $\beta$ is just given by restriction:

$$\mathscr{F}(V) \longrightarrow \mathscr{F}(U\cap V) = j_*(\mathscr{F}\big|_U)(V)\ , \tag{*}$$

and is a morphism of sheaves by presheaf axiom (2).

Now the sequence is exact because the kernel of the above map is exactly those sections $s \in \mathscr{F}(V)$ whose stalks $s_P$ vanish at all $P \notin Z$; by definition this is $\Gamma_{Z\cap V}(V,\mathscr{F}\big|_V) = \mathscr{H}^0_Z(\mathscr{F})(V)$.

Finally, if $\mathscr{F}$ is flasque, then it is clear from (*) that $\beta$ is surjective as a presheaf map, hence as a sheaf map.

2(NC). (10 points) Carefully prove the following lemma (from class on Wednesday, 24 January):

**Lemma.** *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, let*

$$0 \longrightarrow I' \longrightarrow I \longrightarrow I'' \longrightarrow 0$$

*be an exact sequence in $\mathscr{A}$, where $I'$, $I$, and $I''$ are injective, and let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact, covariant functor.*

*Then the sequence*

$$0 \longrightarrow F(I') \longrightarrow F(I) \longrightarrow F(I'') \longrightarrow 0$$

*is also exact*

In proving this, you may use (without proof) properties of abelian categories mentioned in Hartshorne's definition of abelian category, my definition of abelian category from class (originally from Wikipedia, but with additional information), or any other property of abelian categories mentioned in class *prior to* the time when the above lemma was stated in class. However, *do not* use Freyd's Embedding Theorem or any result depending on that theorem.

First, we show that if $\phi\colon A \to B$ is an isomorphism, then it is both monomorphic and epimorphic. Indeed, for all objects $X$, composition with $\phi$ induces bijections $\operatorname{Hom}(X, A) \to \operatorname{Hom}(X, B)$ and $\operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X)$. Therefore both of these maps are injective, so $\phi$ is both monomorphic and epimorphic.

Since $F$ is left exact, the sequence $0 \longrightarrow F(I') \longrightarrow F(I) \longrightarrow F(I'')$ is exact, so it remains only to show that $F(I) \longrightarrow F(I'') \longrightarrow 0$ is exact.

As noted in class, the above short exact sequence splits, so there is a map $i\colon I'' \to I$ such that $p \circ i = 1_{I''}$, where $p\colon I \to I''$ is the second map in the short exact sequence. Applying the functor $F$, we then have that $F(p) \circ F(i) = F(1_{I''}) = 1_{F(I'')}$.

Next, we note that if $\phi\colon A \to B$ and $\psi\colon B \to C$ are maps such that $\psi \circ \phi$ is epimorphic, then $\psi$ is epimorphic. Indeed, for all $X$, composition with $\psi$ and with $\phi$ induce maps

$$\operatorname{Hom}(C, X) \longrightarrow \operatorname{Hom}(B, X) \qquad \text{and} \qquad \operatorname{Hom}(B, X) \longrightarrow \operatorname{Hom}(A, X)\,, \tag{1}$$

respectively, and the composition of these two maps is given by composition with $\psi \circ \phi$. Now since $\psi \circ \phi$ is epimorphic, the composition of the maps (1) is injective; therefore the first of the maps (1) is injective; hence $\psi$ is epimorphic.

Now since $1_{F(I'')}$ is an isomorphism, it is epimorphic; hence $F(p)\colon F(I) \to F(I'')$ is epimorphic.

Then exactness of $F(I) \to F(I'') \to 0$ then follows immediately from the following claim.

If $\phi\colon A \to B$ is an epimorphism, then $A \xrightarrow{\phi} B \longrightarrow 0$ is exact. Indeed, the map $B \longrightarrow 0$ is also epimorphic, since $\operatorname{Hom}(0, X)$ has only one element for all $X$. Therefore the factorizations of both maps into epimorphisms followed by monomorphisms are given by $\phi = 1_B \circ \phi$ and $B \longrightarrow 0 = 1_0 \circ (B \longrightarrow 0)$, respectively (since the isomorphisms $1_B$ and $1_0$ are monomorphic, as noted above). So it suffices to show that $\ker(B \longrightarrow 0) = 1_B$. This is true because $(B \longrightarrow 0) \circ 1_B$ is the zero map, and all maps to $B$ factor uniquely through $1_B$.

3. (10 points) Let $X$ be a noetherian topological space, and let $\mathscr{F}$ be a subsheaf of the constant sheaf $\mathbb{Z}$ on $X$. Show that $\mathscr{F}$ is finitely generated; i.e., that there are open subsets $U_1, \dots, U_n$ of $X$ and sections $s_i \in \mathscr{F}(U_i)$ for all $i$ such that no proper subsheaf of $\mathscr{F}$ contains all of these sections. [**Hint:** Consider the sets $\{P \in X : i \in \mathscr{F}_P\}$ for $i \in \mathbb{Z}$.]

Let $\mathscr{Z}$ denote the constant sheaf $\mathbb{Z}$ on $X$. Note that $\mathscr{Z}_P = \mathbb{Z}$ for all $P \in X$.

Following the hint, for each $i \in \mathbb{Z}$ let $U_i = \{P \in X : \mathscr{F}_P \ni i\}$. This is an open set, because if $P \in U_i$ then $i \in \mathscr{F}_P$ is represented by a pair $(V, s)$, consisting of an open neighborhood $V$ of $P$ in $X$ and a section $s \in \mathscr{F}(V)$ with $s_P = i$. After replacing $V$ with the open subset $s^{-1}(i) \subseteq V$, we may assume that $s$ is the constant function $i$ on $V$. Then $V \subseteq U_i$. This implies that $U_i$ is open.

Since $X$ is noetherian, any nonempty collection of closed subsets of $X$ has a minimal element (otherwise one could construct an infinite descending sequence of closed subsets). Therefore, any nonempty collection of open subsets of $X$ has a maximal element. In particular, the set $\{U_i : i \in \mathbb{Z}_{>0}\}$ has a maximal element $U_m$. Note that if $i \mid j$ then $U_i \subseteq U_j$, so this maximal element is actually a largest element.

We claim that the sections $1 \in \mathscr{F}(U_1), \dots, m \in \mathscr{F}(U_m)$ generate $\mathscr{F}$. Indeed, let $0 \neq j \in \mathscr{F}_P$ for some $P$. This germ is represented by a constant function $j$ on some open subset $V$ containing $P$. We have $V \subseteq U_j \subseteq U_m$, so $m$ is a section of $\mathscr{F}$ on $V$, and therefore $j$ is a multiple of the section $\gcd(j, m)$ on $V$, which is in $\mathscr{F}$ because $V \subseteq U_{\gcd(j,m)}$.