Math 256B. Solutions to Homework 10

1. (10 points) Let X be an integral scheme, let $\mathscr{L}$ be a line sheaf on X, let s be a nonzero rational section of $\mathscr{L}$, and let $D = (s)$. Show that $\mathscr{O}_X(D) \cong \mathscr{L}$.

Let $\{U_i\}$ be a covering of X by nonempty open subsets such that there are isomorphisms $\phi_i \colon \mathscr{L}\big|_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i}$ for all i. Then the collection $\{(U_i, f_i)\}$ represents D, where $f_i = \phi_i(s)$ for all i. These are related on intersections $U_i \cap U_j$ by $\phi_i\big|_{U_i\cap U_j} = (\phi_i(s)/\phi_j(s))\phi_j\big|_{U_i\cap U_j} = (f_i/f_j)\phi_j\big|_{U_i\cap U_j}$ (to see this, compare their values on the rational section s).

The line sheaf $\mathscr{O}(D) \subseteq \mathscr{K}$ is defined by $\mathscr{O}(D)\big|_{U_i} = f_i^{-1}\mathscr{O}_{U_i}$, so we have isomorphisms $\psi_i \colon \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{O}(D)\big|_{U_i}$, defined by $1 \mapsto f_i^{-1}$, for all i. These are related on intersections $U_i \cap U_j$ by $\psi_i\big|_{U_i\cap U_j} = (f_j/f_i)\psi_j\big|_{U_i\cap U_j}$ (to see this, compare their values on the rational section 1_D).

We then have isomorphisms $\psi_i\circ\phi_i \colon \mathscr{L}\big|_{U_i} \xrightarrow{\sim} \mathscr{O}(D)\big|_{U_i}$ for all i. They are related on intersections $U_i\cap U_j$ by $(\psi_i\circ\phi_i)\big|_{U_i\cap U_j} = (f_j/f_i)(f_i/f_j)(\psi_j\circ\phi_j)\big|_{U_i\cap U_j} = (\psi_j\circ\phi_j)\big|_{U_i\cap U_j}$, so they glue to give a well-defined isomorphism $\mathscr{L} \xrightarrow{\sim} \mathscr{O}(D)$.

2. (15 points) Hartshorne II Ex. 6.6. You may ignore the last sentence of (d).

However, note that the definition of the group law on X comes from the group operation on $\operatorname{Cl}^0 X$ and the bijection $\operatorname{Cl}^0 X \to X$ (and not from the line-and-chord operation). Also, make it clear whether you are adding divisors or using the group operation on X.

Let X be the nonsingular plane curve $y^2z = x^3 - xz^2$ of (6.10.2).

(a). Show that three points P, Q, R are collinear if and only if $P + Q + R = 0$ in the group law on X. (Note that the point $P_0 = (0, 1, 0)$ is the zero element of the group structure on X.)

(b). A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.

(c). A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point. (An *inflection point* of a plane curve is a nonsingular point P of the curve, whose tangent line (I, Ex. 7.3) has intersection multiplicity ≥ 3 with the curve at P.)

(d). Let $k = \mathbb{C}$. Show that the points of X with coordinates in $\mathbb{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

(a). First assume that P, Q, and R are collinear, let L be the line containing them, and let $ax + by + cz$ be an element of $k[x, y, z]$ whose vanishing determines L. In the notation of Ex. 6.2a, think of L as a divisor on $\mathbb{P}^2$, and write $L.X = n_1P + n_2Q + n_3R$. Then n_1, n_2, and n_3 are positive integers, and their sum is 3 by Ex. 6.2c (since $\deg X = 3$). Therefore $n_1 = n_2 = n_3 = 1$, and we have $L\,.\,X = P + Q + R$. Also, if

L' is the line $z = 0$, then it meets X only at the point P_0, so again by Ex. 6.2c we have $L' . X = 3P_0$. It then follows that the principal divisor $((ax + by + cz)/z)$ is

$$\left(\frac{ax + by + cz}{z}\right) = P + Q + R - 3P_0 = (P - P_0) + (Q - P_0) + (R - P_0) ,$$

so that $P + Q + R = 0$ in the group law.

This also makes sense if P, Q, and R are not distinct, provided that appropriate multiplicities are used when saying that P, Q, and R are collinear (Ex. 6.2c), so that $L . X = P + Q + R$.

Conversely, let $P + Q + R = 0$ in the group law, and let L be the line joining P and Q. (Or, let L be the tangent line to P if $P = Q$.) Let R' be the other point where L meets X; then by the above, $P + Q + R' = 0$ in the group law. Since also $P + Q + R = 0$, we have $R = R'$. Again, this also makes sense if P, Q, and R are not distinct, if we use the appropriate convention on multiplicities.

(b). We have $2P = 0$ in the group law if and only if $P + P + P_0 = 0$ in the group law; by (a) and (I Ex. 7.3) this holds if and only if the tangent line at P passes through P_0.

(c). We have $3P = 0$ in the group law if and only if $P + P + P = 0$ in the group law; by (a) and Bézout's theorem this holds if and only if the tangent line at P has multiplicity ≥ 3 at P; *i.e.*, if and only if P, P, and P are collinear.

(d). If P and Q have rational coordinates, then so does the additional point R on the line $\overline{PQ}$. Indeed, this is obvious if the line is vertical; otherwise, by the point-slope formula, the slope and y-intercept of the line are rational. Plugging the equation for this line into the equation for X gives a cubic equation in x with rational coefficients. Since two of the roots of this cubic are rational, the third must also be rational and therefore so is the y-coordinate of the third point. Then we form $P + Q$ by taking R and changing the sign of its y coordinate.

(By the Mordell-Weil theorem, this subgroup is a finitely generated abelian group.)

3(NC). (15 points) Let X be a variety over an algebraically closed field k, and let $U = \operatorname{Spec} A$ be a nonempty open affine subset of X.

(a). Let $P \in X$ be a point not in U. Show that there is a function $f \in A$ that does not extend to a regular function at P (i.e., $f \notin \mathscr{O}_{X,P}$). Moreover, f can be chosen from any given generating set for A over k.

[**Hint:** Hartshorne II Ex. 4.2 or Vakil 2023 11.4.A (=Vakil 2017 10.2.A) may be useful (and may be used without proof). But if you use one of them, say which one you are using and how you are using it.]

(b). Assume now that X is a nonsingular curve (not necessarily projective). Choose a closed embedding $i\colon U \to \mathbb{A}^n_k$ over k for some n, let V be the corresponding closed subscheme of $\mathbb{A}^n_k$, and let Y be the closure of V in $\mathbb{P}^n_k$ (with reduced induced subscheme structure). (Here we regard $\mathbb{A}^n_k$ as the open subscheme $D_+(x_0)$ of $\mathbb{P}^n_k$ in the usual way.) As was noted in class, the map $i\colon U \to V$ extends uniquely to a morphism $j\colon X \to Y$. Show that $j^{-1}(V) = U$ (or equivalently, $j^{-1}(Y \setminus V) = X \setminus U$).

(a). It suffices to prove the last assertion (involving the generating set). We will show the contrapositive; i.e., if every element of some given generating set for A over k extends to a regular function at P, then $P \in U$.

Let Σ be a generating set for A over k (it doesn't need to be finite). Assume that $f \in \mathscr{O}_{X,P}$ for all $f \in \Sigma$, or in other words, $\Sigma \subseteq \mathscr{O}_{X,P}$. This implies that $A \subseteq \mathscr{O}_{X,P}$ since A and $\mathscr{O}_{X,P}$ are k-subalgebras of the function field $K(X)$ (this is true because X is integral).

By Hartshorne II Ex. 3.3(c) or Vakil 5.3.3 and 7.3.12, A is of finite type over k. Let $y_1, \dots, y_n$ be a finite generating set for A over k. We may assume that this is a subset of Σ (this is true because each y_i can be written using a finite number of elements of Σ). Each y_i extends to a regular function over an open neighborhood of P in X, so there is an open neighborhood V of P in X such that $A \subseteq \mathscr{O}(V)$ (as k-subalgebras of $K(X)$). We may assume that V is affine, say $V = \operatorname{Spec} B$. Then $A \subseteq B$; let $\phi\colon V \to U$ be the morphism corresponding to the inclusion map $A \hookrightarrow B$.

Since $A \hookrightarrow B$ induces the identity map on $K(X)$, ϕ equals the identity on X as a rational map $X \dashrightarrow X$, so there is a nonempty open subset W of X such that $W \subseteq V$ and $\phi\big|_W$ is the identity map on W. By Hartshorne II Ex. 4.2 or Vakil 11.4.A, ϕ is the identity on V (this uses the fact that X is separated and reduced). Therefore $P = \phi(P) \in \operatorname{Spec} A$, as was to be shown.

This is easier using Vakil's stronger result 11.4.A, as follows. Again, $A \subseteq \mathscr{O}_{X,P}$. Define a map $f\colon \operatorname{Spec} \mathscr{O}_{X,P} \to X$ as a composite $\operatorname{Spec} \mathscr{O}_{X,P} \to \operatorname{Spec} B \to X$, where $\operatorname{Spec} B$ is an open affine in X containing P, and define $g\colon \operatorname{Spec} \mathscr{O}_{X,P} \to X$ by $\operatorname{Spec} \mathscr{O}_{X,P} \to \operatorname{Spec} A \to X$, using $A \hookrightarrow \mathscr{O}_{X,P}$. Let $\mu\colon \operatorname{Spec} K(X) \to \operatorname{Spec} \mathscr{O}_{X,P}$ be the natural map corresponding to $\mathscr{O}_{X,P} \hookrightarrow K(X)$; then (in the language of Vakil 11.4.A) f and g agree on μ. Therefore, by Vakil 11.4.A (since X is separated), there is a closed subscheme $i\colon V \hookrightarrow \operatorname{Spec} \mathscr{O}_{X,P}$ such that μ factors through i, and such that f and g agree on i. Therefore V contains the generic point of $\operatorname{Spec} \mathscr{O}_{X,P}$, so since $\mathscr{O}_{X,P}$ is reduced, $V = \operatorname{Spec} \mathscr{O}_{X,P}$, so $f = g$. This implies $P \in \operatorname{Spec} A$.

(b). We already know that $j(U) \subseteq V$, so it remains only to show that $j(x) \notin V$ for all $x \in X \setminus U$.

Let $x \in X \setminus U$, and suppose $j(x) \in V$. By part (a), there is an element $f \in A$ such that $f \notin \mathscr{O}_{X,x}$. In particular, $A \nsubseteq \mathscr{O}_{X,x}$.

On the other hand, $j^{-1}(V)$ is an open subset of X and it contains x, so there is an open affine neighborhood $W = \operatorname{Spec} B$ of x in $j^{-1}(V)$. Since $i\colon U \to V$ is an isomorphism, the map $W \to V$ corresponds to a ring homomorphism $\phi\colon A \to B$ over k. Moreover, ϕ induces the identity map on $K(X)$. Therefore $A \subseteq B$. But $\mathscr{O}_{X,x}$ is a localization of B; therefore $B \subseteq \mathscr{O}_{X,x}$. Combining these, we have $A \subseteq \mathscr{O}_{X,x}$, a contradiction.

4(NC). (10 points) Hartshorne III Ex. 6.3: Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.

(a). If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \ge 0$.
(b). If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \ge 0$.

We start with an easy lemma.

Lemma. *Let X be an scheme, let $\mathscr{G}$ be a sheaf of $\mathscr{O}_X$-modules, and let $n \in \mathbb{N}$. Then*

$$\mathscr{H}om(\mathscr{O}_X^n, \mathscr{G}) \cong \mathscr{G}^n .$$

In particular, if $\mathscr{G}$ is quasi-coherent or coherent, then $\mathscr{H}om(\mathscr{O}_X^n, \mathscr{G})$ has the same property.

Proof. By Prop. 6.3a,

$$\mathscr{H}om(\mathscr{O}_X^n, \mathscr{G}) \cong \mathscr{H}om(\mathscr{O}_X, \mathscr{G})^n \cong \mathscr{G}^n . \qquad \square$$

By Prop. 6.2, we may assume that X is affine, say $X = \operatorname{Spec} A$. Let M be an A-module such that $\mathscr{F} \cong \widetilde{M}$. Since $\mathscr{F}$ is coherent, M is finitely generated, so there is a left resolution $L. \longrightarrow M$ of M in which each L_i is free of finite rank. Then $\widetilde{L}. \longrightarrow \mathscr{F}$ is a left resolution of $\mathscr{F}$ by free coherent sheaves. By Prop. 6.5,

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \cong h^i(\mathscr{H}om(\widetilde{L}., \mathscr{G})) .$$

Thus:

(a). $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent because it is the cohomology sheaf of a complex of quasi-coherent sheaves (by the lemma), and
(b). $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent because it is the cohomology sheaf of a complex of coherent sheaves.