Math 256B. Solutions to Homework 11

1. (15 points) Hartshorne III Ex. 6.5: Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the *homological dimension* of $\mathscr{F}$, denoted $\operatorname{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:

   (a). $\mathscr{F}$ is locally free $\iff$ $\mathscr{E}xt^1(\mathscr{F},\mathscr{G})=0$ for all $\mathscr{G}\in\mathfrak{Mod}(X)$;
   (b). $\operatorname{hd}(\mathscr{F})\le n$ $\iff$ $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})=0$ for all $i>n$ and all $\mathscr{G}\in\mathfrak{Mod}(X)$;
   (c). $\operatorname{hd}(\mathscr{F})=\sup_x \operatorname{pd}_{\mathscr{O}_{X,x}}\mathscr{F}_x$.

Claim. If $x\in X$ and N is a module over the local ring $\mathscr{O}_{X,x}$, then there exists a sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules such that $\mathscr{G}_x\cong N$.

Proof. Take a skyscraper sheaf. □

(a). If $\mathscr{F}$ is locally free, then $\mathscr{E}xt^1(\mathscr{F},\mathscr{G})=0$ for all $\mathscr{G}$; this follows from Prop. 6.5 using the locally free resolution $0\to\mathscr{F}\to\mathscr{F}\to0$. Conversely, if $\mathscr{E}xt^1(\mathscr{F},\mathscr{G})=0$ for all $\mathscr{G}$, then $\operatorname{Ext}^1_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)=0$ for all $x\in X$ and all $\mathscr{G}\in\mathfrak{Mod}(X)$ by Prop. 6.8; hence (by the claim and Prop. 6.10Aa) $\mathscr{F}_x$ is a projective $\mathscr{O}_{X,x}$-module. Therefore $\mathscr{F}_x$ is a free $\mathscr{O}_{X,x}$-module by Eisenbud Thm. A.3.2, so $\mathscr{F}$ is locally free by Ex. II 5.7b.

(b). (We may assume $n<\infty$.) If $\operatorname{hd}(\mathscr{F})\le n$ then $\mathscr{F}$ has a locally free resolution of length $\le n$; using that resolution to compute $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$, we see that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})=0$ for all $i>n$.

We show the converse statement by induction on n. Suppose $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})=0$ for all $i>n$ and all $\mathscr{G}$. If $n=0$, then (a) implies that $\mathscr{F}$ is locally free, so it has a locally free resolution of length 0; i.e., $0\to\mathscr{F}\to\mathscr{F}\to0$. Thus $\operatorname{hd}(\mathscr{F})\le0$.

Now suppose $n>0$. Let $\mathscr{E}_0$ be a locally free sheaf mapping onto $\mathscr{F}$, and let $\mathscr{K}$ be the kernel:

$$0\longrightarrow\mathscr{K}\longrightarrow\mathscr{E}_0\longrightarrow\mathscr{F}\longrightarrow0\,.$$

Then we have exact sequences

$$\mathscr{E}xt^{i-1}(\mathscr{E}_0,\mathscr{G})\longrightarrow\mathscr{E}xt^{i-1}(\mathscr{K},\mathscr{G})\longrightarrow\mathscr{E}xt^{i}(\mathscr{F},\mathscr{G})\longrightarrow\mathscr{E}xt^{i}(\mathscr{E}_0,\mathscr{G})$$

for all $i>n$. If $i>n>0$, then $i\ge2$, so the first and last terms are zero. Thus $\mathscr{E}xt^{i-1}(\mathscr{K},\mathscr{G})\cong\mathscr{E}xt^{i}(\mathscr{F},\mathscr{G})$, so $\mathscr{E}xt^{i}(\mathscr{K},\mathscr{G})=0$ for all $i>n-1$. By induction, $\operatorname{hd}(\mathscr{K})\le n-1$, so tacking an $\mathscr{E}_0$ onto the end of a suitable locally free resolution of $\mathscr{K}$ gives a locally free resolution of $\mathscr{F}$ showing that $\operatorname{hd}(\mathscr{F})\le n$.

(c). By part (b), the claim, and Prop. 6.10Ab, we have

$$\begin{aligned}\operatorname{hd}(\mathscr{F})&=\sup\{i\in\mathbb{N}:\mathscr{E}xt^i(\mathscr{F},\mathscr{G})\ne0\text{ for some }\mathscr{G}\}\\&=\sup\{i\in\mathbb{N}:\operatorname{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)\ne0\text{ for some }\mathscr{G}\text{ and some }x\in X\}\\&=\sup\{i\in\mathbb{N}:\operatorname{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x,N)\ne0\text{ for some }x\in X\text{ and some }N\}\\&=\sup_{x\in X}\operatorname{pd}_{\mathscr{O}_{X,x}}\mathscr{F}_x\,.\end{aligned}$$

2(NC). (10 points) Hartshorne III Ex. 6.7: Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M and N be A-modules, with M finitely generated. Then

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)$$

and

$$\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)^\sim .$$

For the first part, first note that, since A is noetherian, the category of finitely-generated A-modules (call it $\mathfrak{Noeth}(A)$) is an abelian category.

By II Cor. 5.5 and III Prop. 6.4, the contravariant functors $\mathfrak{Noeth}(A) \to \mathfrak{Ab}$ given by

$$M \mapsto \operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \qquad \text{and} \qquad M \mapsto \operatorname{Ext}^i_A(M, N)$$

are δ-functors, and coincide when $i = 0$ by II Cor. 5.5.

For any $n \in \mathbb{N}$ and $i > 0$, we have

$$\operatorname{Ext}^i_X(\widetilde{A^n}, \widetilde{N}) = \operatorname{Ext}^i_X(\widetilde{A}, \widetilde{N})^n = H^i(X, \widetilde{N}) = 0$$

by Prop. 6.3c and Theorem 3.5, and we also have

$$\operatorname{Ext}^i_A(A^n, N) = \operatorname{Ext}^i_A(A, N)^n = 0$$

because the (covariant) functor $\operatorname{Hom}(A, \cdot)$ is the identity functor, hence is exact.

Since every finitely-generated A-module M admits a surjection from A^n for some n, these δ-functors are coeffaceable. By Theorem 1.3A they therefore coincide for all i. This shows the first half of the problem.

For the second half, we first show:

Lemma. *If M is a finitely-generated A-module and N is any A-module, then*

$$\mathscr{H}om_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Hom}_A(M, N)^\sim .$$

Proof. Since $\Gamma(X, \mathscr{H}om_X(\widetilde{M}, \widetilde{N})) = \operatorname{Hom}_X(\widetilde{M}, \widetilde{N}) = \operatorname{Hom}_A(M, N)$ by II Cor. 5.5, the identity map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N)$ gives a map of sheaves,

$$\operatorname{Hom}_A(M, N)^\sim \to \mathscr{H}om_X(\widetilde{M}, \widetilde{N})$$

by II Ex. 5.3. By Prop. 6.8, we have for all $\mathfrak{p} \in \operatorname{Spec} A = X$ that

$$(\operatorname{Hom}_A(M, N)^\sim)_{\mathfrak{p}} = (\operatorname{Hom}_A(M, N))_{\mathfrak{p}} = \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \mathscr{H}om_X(\widetilde{M}, \widetilde{N})_{\mathfrak{p}} ,$$

so the above map is an isomorphism. □

Now let

$$\ldots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence with $L_i \cong A^{n_i}$ for all $i \in \mathbb{N}$. This gives a locally free resolution of M (viewing this sequence as consisting of modules over the single-point ringed space associated to A). Thus, by Prop. 6.5,

$$\operatorname{Ext}^i_A(M,N) = h^i(\operatorname{Hom}_A(L.,N)) \; .$$

Similarly, $\widetilde{L}. \to \widetilde{M} \to 0$ is a locally free resolution of $\widetilde{M}$ on X, so

$$\begin{aligned} \mathscr{E}xt^i_X(\widetilde{M},\widetilde{N}) &= h^i(\mathscr{H}om_X(\widetilde{L}.,\widetilde{N})) = h^i(\operatorname{Hom}_A(L.,N)\widetilde{\ }) \\ &= h^i(\operatorname{Hom}_A(L.,N))\widetilde{\ } = \operatorname{Ext}^i_A(M,N)\widetilde{\ } \end{aligned}$$

by Prop. 6.5, by the lemma, and by exactness of $\widetilde{\ }$.

3. (10 points) Prove part (c) of Theorem 7.1 without using the theory of δ-functors. Instead, use short and long exact sequences and induction on i, as was done in class in other proofs. Do not worry about showing that the isomorphisms are natural and functorial.

First of all, the case $i = 0$ was already proved in (b), so we may assume that $i > 0$.

As noted in class, by (II, Cor. 5.18) there is a surjection $\mathscr{E} \twoheadrightarrow \mathscr{F}$, where $\mathscr{E}$ is a finite direct sum $\bigoplus \mathscr{O}(-q_j)$ with $q_j > 0$ for all j. Therefore, we have a short exact sequence of coherent sheaves,

$$0 \longrightarrow \mathscr{K} \xrightarrow{\phi} \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0 \; .$$

For all $i > 0$ we have $\operatorname{Ext}^i(\mathscr{E},\omega) \cong \bigoplus \operatorname{Ext}^i(\mathscr{O}_X,\omega(q_j)) \cong \bigoplus H^i(X,\omega(q_j)) = 0$ by Prop. 6.7, by the fact that the direct sum of injective resolutions of a sheaf is an injective resolution of the direct sum of the sheaves, and by Thm. 5.1. Similarly, $H^{n-i}(X,\mathscr{E}) \cong \bigoplus H^{n-i}(X,\mathscr{O}(-q_j)) = 0$ for all $i > 0$ by Thm. 5.1.

By Prop. 6.4 and the dual of the long exact sequence in cohomology, we have exact sequences

$$\operatorname{Ext}^{i-1}(\mathscr{E},\omega) \longrightarrow \operatorname{Ext}^{i-1}(\mathscr{K},\omega) \xrightarrow{f} \operatorname{Ext}^i(\mathscr{F},\omega) \longrightarrow \operatorname{Ext}^i(\mathscr{E},\omega) = 0$$

and

$$H^{n-i+1}(X,\mathscr{E})^\vee \longrightarrow H^{n-i+1}(X,\mathscr{K})^\vee \xrightarrow{g} H^{n-i}(X,\mathscr{F})^\vee \longrightarrow H^{n-i}(X,\mathscr{E})^\vee = 0 \; .$$

Now if $i > 1$ then $\operatorname{Ext}^{i-1}(\mathscr{E},\omega) = H^{n-i+1}(X,\mathscr{E})^\vee = 0$, so f and g are isomorphisms, giving an isomorphism

$$\operatorname{Ext}^i(\mathscr{F},\omega) \xleftarrow{\sim} \operatorname{Ext}^{i-1}(\mathscr{K},\omega) \xrightarrow{\sim} H^{n-i+1}(X,\mathscr{K})^\vee \xrightarrow{\sim} H^{n-i}(X,\mathscr{F})^\vee$$

by induction on i (assuming that we can prove the base case).

For the base case $i = 1$, we have a diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}(\mathscr{E},\omega) & \longrightarrow & \operatorname{Hom}(\mathscr{K},\omega) & \xrightarrow{f} & \operatorname{Ext}^1(\mathscr{F},\omega) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
H^n(X,\mathscr{E})^\vee & \longrightarrow & H^n(X,\mathscr{K})^\vee & \xrightarrow{g} & H^{n-1}(X,\mathscr{F})^\vee & \longrightarrow & 0
\end{array}$$

with exact rows. The two vertical arrows are isomorphisms by Thm. 7.1(b). The square commutes by naturality (in $\mathscr{F}$) of the pairing in the theorem. Indeed, let $\alpha \in \operatorname{Hom}(\mathscr{E},\omega)$. It maps to $\alpha \circ \phi \in \operatorname{Hom}(\mathscr{K},\omega)$, which is then taken to $t \circ H^n(\alpha \circ \phi) \in H^n(X,\mathscr{K})^\vee$, where t is the isomorphism of Thm. 7.1(a). Going around the other way, α is taken to $t \circ H^n(\alpha) \in H^n(X,\mathscr{E})^\vee$, which is taken to $t \circ H^n(\alpha) \circ H^n(\phi) \in H^n(X,\mathscr{K})^\vee$. These are the same element since H^n is a functor, so the diagram commutes.

This gives the desired isomorphism when $i = 1$.