Math 256B. Solutions to Homework 12

1. (10 points) State an analogue of Theorem 7.1 for $n = 0$. Prove it directly, using as few results from Section 7 as possible. (You may use the two definitions of dualizing sheaf and the fact that they are equivalent.)

Theorem. *Let k be a field, let $n = 0$, and let $X = \mathbb{P}^0_k = \operatorname{Spec} k$. Then:*

(a). *From the definition, we have $\omega = \omega_X = \mathscr{O}_X$, since ω_X is $\bigwedge^0$ of a locally free line sheaf of rank 0. Then $H^0(X, \omega) \cong k$ (canonically).*

(b). *For any coherent sheaf $\mathscr{F}$ on X, the natural pairing*

$$\operatorname{Hom}(\mathscr{F}, \omega) \times H^0(X, \mathscr{F}) \longrightarrow H^0(X, \omega) \cong k$$

is a perfect pairing of finite-dimensional vector spaces over k.

(Part (c) is not applicable, because the isomorphism is prescribed when $i = 0$ and is trivial when $i > \dim X = 0$.)

Proof. **(a)**. We have $\omega = \widetilde{k}$, so $H^0(X, \omega) \cong k$.

(b). Since X is affine, $\mathscr{F} \cong \widetilde{V}$, where V is a finitely generated module over k; i.e., a finite-dimensional vector space over k. Also $\operatorname{Hom}(\mathscr{F}, \omega) = \operatorname{Hom}_k(V, k) = V^\vee$. Now, if V and W are finite-dimensional vector spaces over k, then any map $\phi \colon \widetilde{V} \to \widetilde{W}$ comes from a linear transformation $T \colon V \to W$, and this map also induces a map $H^0(X, \widetilde{V}) \to H^0(X, \widetilde{W})$ since $H^0(X, \widetilde{V}) \cong V$ and $H^0(X, \widetilde{W}) \cong W$ canonically, by (II, Cor. 5.5). Therefore the natural pairing in the statement of (b) is just the canonical pairing $V^\vee \times V \to k$. □

2(NC). (10 points) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of a topological space X, and let I' be a subset of I such that $\mathfrak{U}' := (U_i)_{i \in I'}$ also covers X. Construct natural maps $\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^p(\mathfrak{U}', \mathscr{F})$ for all $p \in \mathbb{N}$ and all sheaves $\mathscr{F}$ of abelian groups on X, functorially in $\mathscr{F}$, such that

(i). $\mathscr{C}^\cdot(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^\cdot(\mathfrak{U}', \mathscr{F})$ is a map of complexes for all $\mathscr{F}$, and

(ii). if X is a scheme and if $\mathscr{F}$, $\mathfrak{U}$, and $\mathfrak{U}'$ satisfy the hypotheses of (III, Thm. 4.5), then the maps $\check{H}^p(\mathfrak{U}, \mathscr{F}) \to \check{H}^p(\mathfrak{U}', \mathscr{F})$ induced by this map of complexes are compatible with the maps of (III, Thm. 4.5).

The sheaf $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is a product of sheaves, and $\mathscr{C}^p(\mathfrak{U}', \mathscr{F})$ is a product of a subset of those factors (i.e., the factors corresponding to $i_0 < \dots < i_p$ with $i_j \in I'$ for all j), so define the map $\pi^p \colon \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^p(\mathfrak{U}', \mathscr{F})$ to be the projection to those factors.

To check that it is a map of complexes, recall the formula

$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}\big|_{U_{i_0,\dots,i_{p+1}}} .$$

Here we only need to concern ourselves with factors $(i_0,\dots,i_{p+1})$ with $i_j \in I'$ for all j. But for those factors, all of the terms in the right-hand side involve factors in $\mathscr{C}^p(\mathfrak{U}',\mathscr{F})$, so the projection maps π^p and π^{p+1} commute with the boundary maps d of each complex.

For part (ii), choose an injective resolution $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^\cdot$ of $\mathscr{F}$ in $\mathfrak{Ab}(X)$ and a morphism $\psi^\cdot\colon \mathscr{C}^\cdot(\mathfrak{U}',\mathscr{F}) \to \mathscr{I}^\cdot$ of complexes inducing the identity map on $\mathscr{F}$. Noting that $\pi^\cdot$ also induces the identity on $\mathscr{F}$, it follows that $\phi^\cdot := \psi^\cdot \circ \pi^\cdot \colon \mathscr{C}^\cdot(\mathfrak{U},\mathscr{F}) \to \mathscr{I}^\cdot$ is also a morphism of complexes that induces the identity map on $\mathscr{F}$.

If we use the map $\psi^\cdot \circ \pi^\cdot$ to define the maps $\check{H}^p(\mathfrak{U},\mathscr{F}) \to H^p(X,\mathscr{F})$, then it is clear from functoriality of the cohomology functor h^p that the maps of Lemma 4.4 are compatible with the maps in Čech cohomology induced by $\pi^\cdot$.

3. (10 points) Let k be a field. Compute the dualizing sheaf of the (reduced) scheme

$$X = V(z) \cup V(x,y) \subseteq \mathbb{P}^2_k\ .$$

(This is the disjoint union of a line and a point.)

We start with a general lemma.

Lemma. *Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X. Let U and V be disjoint open subsets of X such that $U \cup V = X$, and let $i\colon U \hookrightarrow X$ and $j\colon V \hookrightarrow X$ be the inclusion maps. Then $\mathscr{F} \cong i_*(\mathscr{F}\big|_U) \times j_*(\mathscr{F}\big|_V)$.*

Proof. For all open subsets W of X, the sheaf axioms give

$$\mathscr{F}(W) \cong \mathscr{F}(W \cap U) \times \mathscr{F}(W \cap V) = \big(i_*(\mathscr{F}\big|_U)\big)(W) \times \big(j_*(\mathscr{F}\big|_V)\big)(W)\ ,$$

compatible with the restriction maps as W varies. □

Let $L = V(z)$ and $P = V(x,y)$ be the line and point, respectively, and let $i\colon L \hookrightarrow X$ and $j\colon P \hookrightarrow X$ be the corresponding closed embeddings. Since $\dim X = 1$, the dualizing sheaf represents the functor $H^1(X,\cdot)^\vee$.

Let $\mathscr{F}$ be a coherent sheaf on X. Then $\mathscr{F}\big|_L$ and $\mathscr{F}\big|_P$ are coherent sheaves on L and P, respectively. Then

$$\begin{aligned} H^1(X,\mathscr{F}) &\cong H^1\Big(X, i_*(\mathscr{F}\big|_L) \oplus j_*(\mathscr{F}\big|_P)\Big) \\ &\cong H^1\big(L,\mathscr{F}\big|_L\big) \oplus H^1\big(P,\mathscr{F}\big|_P\big) \\ &\cong H^1\big(L,\mathscr{F}\big|_L\big)\ , \end{aligned}$$

by the lemma, III Remark 2.9.1, III 2.10, and III Thm. 2.7. This also then holds for the duals, and therefore

$$H^1(X,\mathscr{F})^\vee \cong H^1\big(L,\mathscr{F}\big|_L\big)^\vee \cong \operatorname{Hom}_L\big(\mathscr{F}\big|_L,\omega_L\big) \cong \operatorname{Hom}_X(\mathscr{F}, i_*(\omega_L))\ .$$

Therefore $\omega_X^\circ \cong i_*\mathscr{O}(-2)$.