Math 256B. Solutions to Homework 13

1. (10 points) Let k be a field. Compute the dualizing sheaf of the (reduced) scheme

$$X = V(z) \cup V(x,y) \subseteq \mathbb{P}^3_k .$$

(This is the union of a plane and a line in $\mathbb{P}^3_k$, which intersect at a point.)

Let $P = V(z)$ and $L = V(x,y)$ be the plane and line, and let $i\colon P \hookrightarrow X$ and $j\colon L \hookrightarrow X$ be the corresponding closed embeddings.

Let $\mathscr{F}$ be a coherent sheaf on X. By II 5.8, $i^*\mathscr{F}$ is a coherent sheaf on P. By adjointness of i_* and i^* (p. 110), we have a bijection

$$\operatorname{Hom}_{\mathscr{O}_P}(i^*\mathscr{F}, i^*\mathscr{F}) \cong \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, i_*i^*\mathscr{F}) .$$

Let $\psi\colon \mathscr{F} \to i_*i^*\mathscr{F}$ be the map corresponding to the identity map on $i^*\mathscr{F}$ under this bijection. We claim that ψ is surjective and that the support of its kernel is contained in L. Surjectivity is a local question (on X), so let $U = \operatorname{Spec} A$ be an open affine in X, and let I be the ideal sheaf of $U \cap P$ in U. Then i corresponds to the homomorphism $A \to A/I$. Let M be an A-module such that $\mathscr{F}\big|_U \cong \widetilde{M}$. Then $i_*i^*\mathscr{F} \cong ({}_A(M/IM))^\sim$, and the map $M \to {}_A(M/IM)$ is surjective. Therefore so is ψ. As for the kernel, i is an isomorphism over $X \setminus L$, so ψ is also an isomorphism over this set.

Now let $\mathscr{K} = \ker\psi$, so that we have a short exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \xrightarrow{\psi} i_*i^*\mathscr{F} \longrightarrow 0 .$$

By II 5.8 and II Ex. 5.5, $i_*i^*\mathscr{F}$ is coherent; therefore so is $\mathscr{K}$. From the corresponding long exact sequence in cohomology, we have

$$H^2(X,\mathscr{K}) \longrightarrow H^2(X,\mathscr{F}) \longrightarrow H^2(X, i_*i^*\mathscr{F}) \longrightarrow H^3(X,\mathscr{K}) .$$

Since $\operatorname{Supp}\mathscr{K} \subseteq L$ has dimension < 2, $H^2(X,\mathscr{K}) = H^3(X,\mathscr{K}) = 0$, so the middle map is an isomorphism. Therefore,

$$H^2(X,\mathscr{F}) \cong H^2(X, i_*i^*\mathscr{F}) \cong H^2(P, i^*\mathscr{F}) \cong \operatorname{Hom}_P(i^*\mathscr{F}, \omega_P)^\vee \cong \operatorname{Hom}_X(\mathscr{F}, i_*\omega_P)^\vee .$$

Here the third isomorphism is by duality (since $i^*\mathscr{F}$ is coherent), and the fourth follows from adjointness of i^* and i_*.

Therefore $\omega_X^\circ \cong i_*\omega_P \cong i_*\mathscr{O}(-3)$.

2. (15 points) Prove that the derived-functor cohomology class determined by the Čech cocycle α of (III, Remark 7.1.1) is invariant under automorphisms of $\mathbb{P}^n_k$. Do this by direct computation.

   [**Hints:** You may find (II, Example 7.1.1) and Exercise 2 on Homework 12 useful. Remember elementary matrices. For information on differentials, see pages 172–173 and (II, 8.13) in Hartshorne.]

   Let k be a field, let $n \in \mathbb{Z}_{>0}$, and let $X = \mathbb{P}^n_k$.

   We start with a lemma that will be used twice in this solution.

Lemma. *Let $I = \{0, 1, \dots, n\}$, let $\tau \in \mathbb{R} \setminus I$, let $I'' = I \cup \{\tau\}$, let $\sigma \in I$, and let $I' = I'' \setminus \{\sigma\}$. Let U_i be open affine subsets of X for all $i \in I''$. Let $\mathfrak{U} = (U_i)_{i\in I}$, $\mathfrak{U}' = (U_i)_{i\in I'}$, and $\mathfrak{U}'' = (U_i)_{i\in I''}$. Assume that $\mathfrak{U}$ and $\mathfrak{U}'$ (and therefore also $\mathfrak{U}''$) cover X, so that they all satisfy the conditions of (III, Thm. 4.5). Let $U_I = \bigcap_{i\in I} U_i$, $U_{I'} = \bigcap_{i\in I'} U_i$, and $U_{I''} = \bigcap_{i\in I''} U_i$. Finally, let $\mathscr{F}$ be a quasi-coherent sheaf on X and let $\alpha \in \Gamma(U_I, \mathscr{F})$ and $\alpha' \in \Gamma(U_{I'}, \mathscr{F})$ be sections. Then*

(a). *the above sections α and α' are cocycles in $C^n(\mathfrak{U}, \mathscr{F})$ and $C^n(\mathfrak{U}', \mathscr{F})$, respectively; and*

(b). *if there exists a cocycle $\beta \in C^n(\mathfrak{U}'', \mathscr{F})$ whose components β_I and $\beta_{I'}$ are the above sections α and α', respectively, then the Čech cohomology classes of α and α' are mapped to the same element in $H^n(X, \mathscr{F})$ by the relevant maps of (III, Lemma 4.4).*

Proof. First of all, since I has $n+1$ elements, $C^n(\mathfrak{U}, \mathscr{F})$ is just $\Gamma(U_I, \mathscr{F})$, so (by slight abuse of notation) we may regard α as an element of $C^n(\mathfrak{U}, \mathscr{F})$. It is a cocycle because $C^{n+1}(\mathfrak{U}, \mathscr{F}) = 0$ (it is an empty product). The same is true for α'. This proves (a).

Next, we consider (b). By Question 2 on Homework 12, there are maps

$$\pi^\cdot : \mathscr{C}^\cdot(\mathfrak{U}'', \mathscr{F}) \to \mathscr{C}^\cdot(\mathfrak{U}, \mathscr{F}) \qquad \text{and} \qquad \pi'^\cdot : \mathscr{C}^\cdot(\mathfrak{U}'', \mathscr{F}) \to \mathscr{C}^\cdot(\mathfrak{U}', \mathscr{F}) \tag{1}$$

of complexes, which give the diagram

$$\begin{array}{ccc} \check{H}^n(\mathfrak{U}'', \mathscr{F}) & \longrightarrow & \check{H}^n(\mathfrak{U}, \mathscr{F}) \\ \big\downarrow & & \big\downarrow \\ \check{H}^n(\mathfrak{U}', \mathscr{F}) & \longrightarrow & H^n(X, \mathscr{F}) . \end{array} \tag{2}$$

The above diagram commutes, because the corresponding diagram of maps of complexes commutes up to homotopy, by what was proved early in the semester about maps of complexes to injective resolutions.

Since the map of (III, Lemma 4.4) is obtained by applying $\Gamma(X, \cdot)$ to the Čech complexes of sheaves (giving the Čech complexes $C^\cdot(\mathscr{U}, \mathscr{F})$ of abelian groups), and then applying h^n, and since $\beta \in C^n(\mathfrak{U}'', \mathscr{F})$ is mapped to $\alpha \in C^n(\mathfrak{U}, \mathscr{F})$ and

$\alpha' \in C^n(\mathfrak{U}', \mathscr{F})$ by the maps on global sections obtained from the maps of (1), we obtain that the cohomology classes of α and α' are mapped to the same element in $H^n(X, \mathscr{F})$, as was to be shown (because they are both mapped to the image of the cohomology class of β in $H^n(X, \mathscr{F})$, by commutativity of the diagram (2)). □

Now we consider the problem at hand.

Let $x_0, \dots, x_n$ be the standard homogeneous coordinates on $X = \mathbb{P}^n_k$, let $\omega = \omega_X$, and let α be as in (III, Remark 7.1.1). Let $y_i = x_i/x_0$ for all $i = 1, \dots, n$. Then

$$\alpha = \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} . \tag{3}$$

Let c be the cohomology class in $H^n(X, \omega)$ corresponding to α (under the map of Lemma 4.4, with $\mathfrak{U}$ as in 7.1.1).

When applying the lemma, we will always let $U_i = D_+(x_i)$ be the standard open subset for all $i \in I$, so that $\mathfrak{U}$ is the standard open cover of $X = \mathbb{P}^n_k$. Also, we will use $\mathscr{F} = \omega$, we will let α be as in (3), and we will use the ordering on I'' inherited from the ordering on $\mathbb{R}$.

By (II, Example 7.1.1), all automorphisms of $\mathbb{P}^n_k$ can be expressed using matrices in $\mathrm{GL}_{n+1}(k)$, and by Gaussian elimination every such matrix is a product of elementary matrices. Therefore it will suffice to show that $\phi^* c = c$ for each of the following types of automorphisms ϕ.

(1) ϕ interchanges the variables x_j and x_ℓ (with $j \ne \ell$ and $j, \ell \in \{0, 1, \dots, n\}$). Since any permutation can be accomplished by successively interchanging consecutive elements, we may assume that $\ell = j + 1$, so ϕ interchanges x_j and x_{j+1}, where $0 \le j < n$.

We start by showing that $\phi^*\alpha = -\alpha$.

Case I. Assume that $j > 0$. Then we are interchanging y_j and y_{j+1}, so $\phi^*\alpha = -\alpha$ is immediate from the fact that the wedge product anticommutes.

Case II. Assume that $j = 0$. Then we are interchanging x_0 and x_1.

This changes y_1 to $1/y_1$ and y_i to y_i/y_1 when $i > 1$. Note that

$$\frac{d(uv)}{uv} = \frac{u\,dv + v\,du}{uv} = \frac{du}{u} + \frac{dv}{v} ,$$

so du/u acts like $d \log u$, even though there is no logarithm function on $K(\mathbb{P}^n_k)$. So

$$\phi^*\alpha = \frac{d(1/y_1)}{1/y_1} \wedge \bigwedge_{i=2}^{n} \frac{d(y_i/y_1)}{y_i/y_1} = -\frac{dy_1}{y_1} \wedge \bigwedge_{i=2}^{n} \left(\frac{dy_i}{y_i} - \frac{dy_1}{y_1} \right) = -\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} = -\alpha ,$$

as was to be shown.

To apply the lemma, let $\sigma = j$, let $\tau = j+1.5$, and let $U_\tau = U_j$. Then both $\mathfrak{U}$ and $\mathfrak{U}'$ will be the standard open covering of $\mathbb{P}^n_k$, but the ordering on $I' = \{0, 1, \dots, j-1,$

$j+1, \tau, j+2, \dots, n\}$ will reflect the fact that U_j and U_{j+1} have been interchanged. In other words, $\mathfrak{U}' = \phi^*\mathfrak{U}$ (including the ordering). Note also that

$$I'' = \{0, 1, \dots, j, j+1, \tau, j+2, \dots, n\}$$

(where the list of elements reflects the ordering).

Let $\alpha' = \phi^*\alpha = -\alpha \in C^n(\mathfrak{U}', \omega)$, and let β be the element of $C^n(\mathfrak{U}'', \omega)$ such that $\beta_I = \alpha$, $\beta_{I'} = \alpha'$, and all other components of β are zero. Since $C^{n+1}(\mathfrak{U}'', \omega)$ has only one factor $\Gamma(U_{I''}, \omega)$, and since $I = I'' \setminus \{j+1.5\}$ and $I' = I'' \setminus \{j\}$, we have

$$d\beta = (-1)^{j+2}\alpha + (-1)^j\alpha' = (-1)^j(\alpha + \alpha') = 0 .$$

(Note that $U_{I''} = U_{I'} = U_I$ due to the duplication $U_\tau = U_j$, so restriction operations are not necessary.)

Thus β is a cocycle in $C^{\cdot}(\mathfrak{U}'', \omega)$, and the lemma then implies that $[\phi^*\alpha] = [\alpha]$ in $H^n(X, \omega)$.

(2) $\phi\colon [x_0 : \dots, x_n] \mapsto [x_0 + \lambda x_1 : x_1 : x_2 : \dots : x_n]$ with $\lambda \in k$.

We have

$$\begin{aligned}
\phi^*\alpha &= \frac{d\left(\frac{x_1}{x_0+\lambda x_1}\right)}{\frac{x_1}{x_0+\lambda x_1}} \wedge \frac{d\left(\frac{x_2}{x_0+\lambda x_1}\right)}{\frac{x_2}{x_0+\lambda x_1}} \wedge \dots \wedge \frac{d\left(\frac{x_n}{x_0+\lambda x_1}\right)}{\frac{x_n}{x_0+\lambda x_1}} \\
&= \bigwedge_{i=1}^n \left(\frac{d\left(\frac{x_i}{x_0}\right)}{\frac{x_i}{x_0}} - \frac{d\left(\frac{x_0+\lambda x_1}{x_0}\right)}{\frac{x_0+\lambda x_1}{x_0}} \right) \\
&= \bigwedge_{i=1}^n \left(\frac{dy_i}{y_i} - \frac{d(\lambda y_1 + 1)}{\lambda y_1 + 1} \right) \\
&= \left(\frac{1}{y_1} - \frac{\lambda}{\lambda y_1 + 1} \right) dy_1 \wedge \bigwedge_{i=2}^n \frac{dy_i}{y_i} \\
&= \frac{1}{\lambda y_1 + 1} \bigwedge_{i=1}^n \frac{dy_i}{y_i} .
\end{aligned}$$

This is not equal to $\pm\alpha$, so more work is required. We need to handle the difference

$$\begin{aligned}
\gamma := \alpha - \phi^*\alpha &= \left(1 - \frac{1}{\lambda y_1 + 1}\right) \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_n}{y_n} \\
&= \frac{\lambda y_1}{\lambda y_1 + 1} \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_n}{y_n} .
\end{aligned}$$

Note that the factor y_1 in the above expression cancels, so γ extends to a global section of ω over $D_+(x_0 + \lambda x_1) \cap D_+(x_0) \cap D_+(x_2) \cap \dots \cap D_+(x_n)$. Therefore, we can apply the lemma as follows.

Let $\tau = -1$, so that $I'' = \{-1, 0, \dots, n\}$, let $U_{-1} = D_+(x_0 + \lambda x_1)$, and let $\sigma = 0$. Let $\alpha' = \phi^*\alpha \in C^n(\mathfrak{U}', \omega)$ (this can be seen by looking at the first expression for $\phi^*\alpha$ above). As noted above, we have $\gamma \in \Gamma(U_{I''\setminus\{1\}}, \omega)$.

Thus, we may let β be the element of $C^n(\mathfrak{U}'', \omega)$ such that $\beta_I = \alpha$, $\beta_{I'} = \alpha'$, $\beta_{I''\setminus\{1\}} = -\gamma$, and all other components of β are zero. Since $C^{n+1}(\mathfrak{U}'', \omega)$ has only one factor $\Gamma(U_{I''}, \omega)$, we have

$$d\beta = \beta_I - \beta_{I'} + \beta_{I''\setminus\{1\}} = \alpha - \alpha' + (-\gamma) = 0 .$$

Thus, we again have that $[\alpha]$ and $[\phi^*\alpha]$ give the same element of $H^n(X, \omega)$.

(3) $\phi\colon [x_0 : x_1 : x_2 : \dots : x_n] \mapsto [x_0 : \lambda x_1 : x_2 : \dots : x_n]$ with $\lambda \in k^*$. Here α is invariant because y_1 is multiplied by λ and all other y_i are unchanged, and $d(\lambda y_1)/(\lambda y_1) = dy_1/y_1$. Since the open cover is also the same (and with the same ordering), the cohomology class c is unchanged.

3(NC). (10 points) Let k be a field, let $n \in \mathbb{Z}_{>0}$, and let $X = \mathbb{P}^n_k$. Conclude from the previous exercise that there is a *canonical* trace map

$$t\colon H^n(X, \omega_X) \to k .$$

By (III, Thm. 7.1a), we have $H^n(X, \omega_X) \cong k$, so by the previous exercise it will suffice to show that the cohomology class of α is nonzero, because then t can be chosen such that $t([\alpha]) = 1$.

This is done using Čech cohomology, either via an isomorphism $\omega_X \cong \mathscr{O}(-n-1)$, or directly in ω_X.

Proof using an isomorphism $\omega_X \xrightarrow{\sim} \mathscr{O}(-n-1)$. Let $\mathfrak{U} = \{U_0, \dots, U_n\}$ be the covering used in the proof of (III, 5.1), where $U_i = D_+(x_i)$ for all i. As noted in the proof of (III, 5.1c), the Čech cohomology group $H^n(\mathfrak{U}, \mathscr{O}(-n-1))$ is generated by $x_0^{-1} \cdots x_n^{-1} \in \Gamma(U_0 \cap \dots \cap U_n, \mathscr{O}(-n-1)) = S(-n-1)_{(x_0 \cdots x_n)}$.

In order to relate this to ω_X, we start by constructing an explicit isomorphism $\omega_X \xrightarrow{\sim} \mathscr{O}(-n-1)$. This is equivalent to giving an isomorphism $\omega_X(n+1) \xrightarrow{\sim} \mathscr{O}_X$, which in turn is equivalent to giving a nonzero global section of $\omega_X(n+1)$.

This isomorphism does not need to be canonical (in fact, it can't be), but it does need to be global.

The set U_0 is affine, equal to $\operatorname{Spec} k[y_1, \dots, y_n]$, where $y_i = x_i/x_0$ ($i = 1, \dots, n$). Then the section $dy_1 \wedge \dots \wedge dy_n$ generates ω_X over U_0, and therefore $x_0^{n+1} dy_1 \wedge \dots \wedge dy_n$ generates $\omega_X(n+1)$ over U_0.

Now consider what happens on U_n. This set is affine, equal to $\operatorname{Spec} k[z_0, \dots, z_n]$, where $z_i = x_i/x_n$ ($i = 0, \dots, n-1$), and $\omega_X(n+1)$ is generated over U_n by $x_n^{n+1} dz_0 \wedge \dots \wedge dz_{n-1}$. Since $y_i = z_i/z_0$ for $1 \le i < n$ and $y_n = 1/z_0$, we have

$$dy_i = \frac{dz_i}{z_0} - \frac{z_i dz_0}{z_0^2} \quad (i = 1, \dots, n-1) \qquad \text{and} \qquad dy_n = -\frac{dz_0}{z_0^2} ,$$

and therefore

$$\begin{aligned} x_0^{n+1} dy_1 \wedge \cdots \wedge dy_n &= x_0^{n+1} \left(\frac{dz_1}{z_0} - \frac{z_1 dz_0}{z_0^2} \right) \wedge \cdots \wedge \left(\frac{dz_{n-1}}{z_0} - \frac{z_{n-1} dz_0}{z_0^2} \right) \wedge \left(-\frac{dz_0}{z_0^2} \right) \\ &= x_0^{n+1} \left(\frac{dz_1}{z_0} \right) \wedge \cdots \wedge \left(\frac{dz_{n-1}}{z_0} \right) \wedge \left(-\frac{dz_0}{z_0^2} \right) \\ &= (-1)^n \left(\frac{x_0^{n+1}}{z_0^{n+1}} \right) dz_0 \wedge dz_1 \wedge \cdots \wedge dz_{n-1} \\ &= (-1)^n x_n^{n+1} dz_0 \wedge dz_1 \wedge \cdots \wedge dz_{n-1} \ , \end{aligned}$$

which, as noted above, is a generator for $\omega_X(n+1)$ over U_n.

Therefore the map is an isomorphism over $U_0 \cup U_n$. By similar arguments, the map is an isomorphism over U_i for all i, so in fact we have an isomorphism $\omega_X(n+1) \xrightarrow{\sim} \mathscr{O}_X$, as desired.

This isomorphism takes

$$\alpha = \frac{x_0^n}{x_1 \cdots x_n} dy_1 \wedge \cdots \wedge dy_n = \frac{x_0^{n+1} dy_1 \wedge \cdots \wedge dy_n}{x_0 x_1 \cdots x_n}$$

to

$$\frac{1}{x_0 x_1 \cdots x_n} \ ,$$

which generates $H^n(X, \mathscr{O}(-n-1))$. Therefore α generates $H^n(X, \omega_X)$, as was to be shown. □

Proof by working directly on ω_X. As before, we let $y_j = x_j/x_0$ for all $j = 1, \ldots, n$, so that $U_0 = \operatorname{Spec} k[y_1, \ldots, y_n]$ and

$$\begin{aligned} \alpha &= \frac{x_0^n}{x_1 \cdots x_n} d\left(\frac{x_1}{x_0} \right) \wedge \cdots \wedge d\left(\frac{x_n}{x_0} \right) \\ &= y_1^{-1} \cdots y_n^{-1} dy_1 \wedge \cdots \wedge dy_n \in C^n(\mathfrak{U}, \omega_X) = \Gamma(U_0 \cap \cdots \cap U_n, \omega_x) \ . \end{aligned}$$

Showing that the cohomology class of α in $\check{H}^n(X, \omega_X)$ is nonzero is then a matter of showing that α is not in the image of $C^{n-1}(\mathfrak{U}, \omega_X) \to C^n(\mathfrak{U}, \omega_X)$.

In what follows, we will use multiindex notation, so that $y^{\mathbf{i}} = y_1^{i_1} \cdots y_n^{i_n}$, where $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$. Then the set

$$B := \{ y^{\mathbf{i}} dy_1 \wedge \cdots \wedge dy_n : \mathbf{i} \in \mathbb{Z}^n \}$$

is a basis for $C^n(\mathfrak{U}, \omega_X)$. We also let $U_{\widehat{\jmath}}$ denote $U_0 \cap \cdots \cap \widehat{U}_j \cap \cdots \cap U_n$ for all $j = 0, \ldots, n$, where the hat denotes omission, so that

$$C^{n-1}(\mathfrak{U}, \omega_X) = \prod_{j=0}^{n} \Gamma(U_{\widehat{\jmath}}, \omega_X) \ .$$

Therefore

$$\operatorname{im}(C^{n-1}(\mathfrak{U},\omega_X)\to C^n(\mathfrak{U},\omega_X))=\sum_{j=0}^{n}\operatorname{im}(\Gamma(U_{\widehat{\jmath}},\omega_X)\to C^n(\mathfrak{U},\omega_X))\ .$$

Now, for all $j>0$, $U_{\widehat{\jmath}}\subseteq U_0$, so the image of $\Gamma(U_{\widehat{\jmath}},\omega_X)\to C^n(\mathfrak{U},\omega_X)$ can be computed using the generator $dy_1\wedge\cdots\wedge dy_n$ of ω_X over U_0. This image is spanned by the subset

$$B_j:=\{y^{\mathbf{i}}dy_1\wedge\cdots\wedge dy_n:\mathbf{i}\in\mathbb{Z}^n,\ i_j\ge 0\}\subseteq B\ .$$

To compute B_0, we note that $U_{\widehat{0}}\subseteq U_n$, and U_n has affine coordinates $z_0,\dots,z_{n-1}$, where $z_0=1/y_n$ and $z_i=y_i/y_n$ for all $0<i<n$. Then $dz_0=-dy_n/y_n^2$ and $dz_i=dy_i/y_n-y_idy_n/y_n^2$, and therefore

$$dz_0\wedge\cdots\wedge dz_{n-1}=-\frac{dy_n}{y_n^2}\wedge\frac{dy_1}{y_n}\wedge\cdots\wedge\frac{dy_{n-1}}{y_n}=(-1)^n\frac{dy_1\wedge\cdots\wedge dy_n}{y_n^{n+1}}\ .$$

Therefore the image of $\Gamma(U_{\widehat{0}},\omega_X)\to C^n(\mathfrak{U},\omega_X)$ is spanned by the set

$$B_0:=\left\{z_0^{j_0}\cdots z_{n-1}^{j_{n-1}}\frac{dy_1\wedge\cdots\wedge dy_n}{y_n^{n+1}}:\mathbf{j}\in\mathbb{Z}^n,\ j_0\ge 0\right\}$$

Since $z_0^{j_0}\cdots z_{n-1}^{j_{n-1}}/y_n^{n+1}=y_1^{j_1}\cdots y_{n-1}^{j_{n-1}}y_n^{-(j_0+\cdots+j_{n-1}+n+1)}=\mathbf{y}^{\mathbf{i}}$, where $(i_1,\dots,i_n)=(j_1,\dots,j_{n-1},-j_0-\cdots-j_{n-1}-n-1)$ and therefore $j_0=-i_1-\cdots-i_n-n-1$, this set is

$$B_0=\{y^{\mathbf{i}}dy_1\wedge\cdots\wedge dy_n:\mathbf{i}\in\mathbb{Z}^n,\ i_1+\cdots+i_n\le -n-1\}\subseteq B\ .$$

Since all of the sets B_j are subsets of B, and none contain α (which corresponds to $i_1=\cdots=i_n=-1$), we have that α is not in the image of $C^{n-1}(\mathfrak{U},\omega_X)\to C^n(\mathfrak{U},\omega_X)$, as was to be shown. □

(For those of you familiar with duality of de Rham cohomology in analysis, our trace map corresponds to the integration map of n-forms on a real n-manifold.)