Math 256B. Solutions to Homework 14

1(NC). (10 points) Let k be a field. Find the dualizing sheaf of $V(xy)$ in $\mathbb{P}^2_k$ (the union of two lines in $\mathbb{P}^2_k$ intersecting in a point, with reduced induced subscheme structure).

Either express it as the restriction to X of a line sheaf $\mathscr{L}$ on $\mathbb{P}^2_k$, or show that no such $\mathscr{L}$ exists.

[**Hint:** Don't work too hard.]

Since X is a Cartier divisor, cut out by the homogeneous polynomial xy of degree 2, its ideal sheaf is $\mathscr{I}_X \cong \mathscr{O}(-X) \cong \mathscr{O}(-2)$. Since X is also a complete intersection, III Thm. 7.11 gives

$$\omega_X^\circ \cong \omega_P \otimes (\mathscr{I}_X/\mathscr{I}_X^2)^\vee \cong (\mathscr{O}(-3) \otimes \mathscr{O}(-2)^\vee)\big|_X \cong \mathscr{O}(-1)\big|_X .$$

2. (10 points) Let A be a ring. Show that $\operatorname{Tor}_i^A(M,N) \cong \operatorname{Tor}_i^A(N,M)$ for all A-modules M and N, and for all $i \in \mathbb{N}$ (without looking it up anywhere). Use a spectral sequence.

[**Hint:** Use the opposite(s) of one or more categories.]

This proof will be modeled after the proof of Prop. 6.5 given in class. However, Tor is a left derived functor, so things need to be done a bit differently here. We would need to either develop a theory of spectral sequences for double complexes in the third quadrant, or reverse all of the arrows.

We choose the latter (reversing the arrows). This is possible because the definition of abelian category is symmetric in reversal of the arrows, so in particular the opposite of an abelian category is also an abelian category.

But first, we describe Tor. Let A be a ring. Classically, $\operatorname{Tor}_i^A(M,\cdot)$ is the i^{th} left derived functor of the (right exact) functor $M \otimes_A \cdot$, from the category $\mathfrak{Mod}(A)$ to itself. The category $\mathfrak{Mod}(A)$ has enough free modules (of arbitrary rank), hence enough projectives, so this works.

We let $\mathfrak{Dom}(A)$ denote the opposite of the category $\mathfrak{Mod}(A)$ (*Dom* is *Mod* spelled backwards – get it?). As noted above, this is an abelian category. Then $M \otimes_A \cdot$ is a covariant left exact functor from $\mathfrak{Dom}(A)$ to itself, and the i^{th} right derived functors give the objects $\operatorname{Tor}_i^A(M,\cdot)$, with the corresponding morphisms reversed in direction. In the category $\mathfrak{Dom}(A)$, free modules are injective, so a free resolution $F_\cdot \to N \to 0$ of an A-modules N (in $\mathfrak{Mod}(A)$) gives an injective resolution $0 \to N \to I^\cdot$ in $\mathfrak{Dom}(A)$, where $I^i = F_i$ for all i. Therefore, $\operatorname{Tor}_i(M,N) = h^i(M \otimes_A I^\cdot)$.

Now fix A-modules M and N, and let $0 \to M \to I^\cdot$ and $0 \to N \to J^\cdot$ be injective resolutions of M and N, respectively, in $\mathfrak{Dom}(A)$ (coming from free, hence projective, resolutions in $\mathfrak{Mod}(A)$). Define a double complex $K^{\cdot,\cdot}$ by

$$K^{p,q} = I^p \otimes_A J^q \qquad \text{for all } p,q \in \mathbb{N} ,$$

with differentials $d^{p,q}\colon K^{p,q} \to K^{p+1,q}$ and $\delta^{p,q}\colon K^{p,q} \to K^{p,q+1}$ determined by the maps $I^p \to I^{p+1}$ and $J^q \to J^{q+1}$, respectively.

Let $\{E_r^{\cdot,\cdot}\}_{r\in\mathbb{N}}$ be the spectral sequence obtained from this double complex. Then (as was noted in class on April 24), $E_0^{p,q} \cong K^{p,q} \cong I^p \otimes_A J^q$, $d_0^{p,q}\colon E_0^{p,q} \to E_0^{p,q+1}$ is the map $\delta^{p,q}$, and

$$E_1^{p,q} = h^q(K^{p,\cdot}) = h^q(I^p \otimes_A J^\cdot)\ .$$

Since all I^p are free modules,

$$h^q(I^p \otimes_A J^\cdot) \cong I^p \otimes_A h^q(J^\cdot) \cong \begin{cases} I^p \otimes_A N & \text{if } q = 0\ ; \\ 0 & \text{if } q > 0\ . \end{cases}$$

Again, from class on April 22–24, the maps $d_1^{p,0}\colon E_1^{p,0} \to E_1^{p+1,0}$ are the maps in cohomology induced by $\delta^{p,\cdot}\colon K^{p,\cdot} \to K^{p+1,\cdot}$, so they are the maps $I^p \otimes_A N \to I^{p+1} \otimes_A N$. The E_2 terms of this spectral sequence are then the cohomology of the E_1 terms in d_1; hence

$$E_2^{p,0} = h^p(I^\cdot \otimes_A N) = \operatorname{Tor}_p^A(N, M)\ ,$$

and $E_2^{p,q} = 0$ for all $q \ne 0$. Therefore, as was done in class,

$$H^p(K^\cdot) \cong E_2^{p,0} \cong \operatorname{Tor}_p^A(N, M) \qquad \text{for all } p\ .$$

Now let $\widetilde{K}^{\cdot,\cdot}$ be the transpose of the double complex $K^{\cdot,\cdot}$ (i.e., $\widetilde{K}^{p,q} = K^{q,p}$, and d and δ are interchanged accordingly). A computation similar to the above gives

$$H^p(\widetilde{K}^\cdot) \cong \operatorname{Tor}_p^A(M, N) \qquad \text{for all } p\ .$$

As was noted in class, we have $H^p(\widetilde{K}^\cdot) \cong H^p(K^\cdot)$; hence

$$\operatorname{Tor}_p^A(M, N) \cong \operatorname{Tor}_p^A(N, M) \qquad \text{for all } p\ .$$

3(NC). (10 points) Hartshorne III Ex. 8.1: Let $f\colon X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_* \mathscr{F} = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \ge 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})\ .$$

(This is a degenerate case of the Leray spectral sequence—see Godement [1, II, 4.17.1].)

Let $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ be a flasque resolution of $\mathscr{F}$. By Cor. 8.3, this resolution can be used to compute $R^i f_* \mathscr{F}$, so

$$h^i(f_*\mathscr{I}^\cdot) = \begin{cases} f_*\mathscr{F} & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Thus $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\cdot$ is exact; since the $f_*\mathscr{I}^i$ are flasque for all i (II Ex. 1.16d), this is a flasque resolution of $f_*\mathscr{F}$. We can use these flasque resolutions to compute cohomology, so:

$$H^i(X, \mathscr{F}) \cong h^i(\Gamma(X, \mathscr{I}^\cdot)) = h^i(\Gamma(Y, f_*\mathscr{I}^\cdot)) \cong H^i(Y, f_*\mathscr{F})\ .$$

4. (20 points) Hartshorne III Ex. 8.4: Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank $n+1$, $n \ge 1$. Let $X = \mathbb{P}(\mathscr{E})$ (II, §7), with the invertible sheaf $\mathscr{O}_X(1)$ and the projection morphism $\pi\colon X \to Y$.
   (a). Then $\pi_*\mathscr{O}(l) \cong S^l(\mathscr{E})$ for $l \ge 0$, $\pi_*\mathscr{O}(l) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathscr{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbb{Z}$; and $R^n\pi_*(\mathscr{O}(l)) = 0$ for $l > -n-1$.
   (b). Show that there is a natural exact sequence

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow (\pi^*\mathscr{E})(-1) \longrightarrow \mathscr{O} \longrightarrow 0\,,$$

   cf. (**, 8.13), and conclude that the *relative canonical sheaf* $\omega_{X/Y} = \wedge^n\Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathscr{O}_Y$ (cf. (7.1.1)).
   (c). Now show, for any $l \in \mathbb{Z}$, that

$$R^n\pi_*(\mathscr{O}(l)) \cong \pi_*(\mathscr{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee\,.$$

   (d). Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).
   (e). In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a *geometrically ruled surface* (V, §2).

For part (b), you may assume that Remark 7.1.1 is true for arbitrary commutative rings. For part (d), assume that Y is a nonsingular variety.

Parts (a)–(c) basically give a relative version of (III, Thm. 5.1).

Some of you had problems with going from the local case to the global case. It's best to start with as many global things as possible, and then prove various properties locally.

(a). By II 7.11 and the proof of II 5.13, the natural map $\bigoplus_{l\in\mathbb{N}} S^l\mathscr{E} \to \bigoplus_{l\in\mathbb{Z}} \pi_*\mathscr{O}(l)$ is an isomorphism. This gives $\pi_*\mathscr{O}(l) \cong S^l(\mathscr{E})$ for $l \ge 0$ and $\pi_*\mathscr{O}(l) = 0$ for all $l < 0$. In the special case $Y = \operatorname{Spec} A$ with A noetherian, we have $R^i\pi_*(\mathscr{O}(l)) = 0$ for $0 < l < n$ and $R^n\pi_*(\mathscr{O}(l)) = 0$ for $l > -n-1$ by Prop. 8.5 and Thm. 5.1. The general case then follows by Cor. 8.2. All of these isomorphisms are natural.

(b). It suffices to show instead that there is a natural short exact sequence

$$0 \longrightarrow \Omega_{X/Y}(1) \xrightarrow{\alpha} \pi^*\mathscr{E} \xrightarrow{\beta} \mathscr{O}_X(1) \longrightarrow 0\,. \tag{*}$$

Here β is the natural map $\pi^*\pi_*\mathscr{O}_X(1) \to \mathscr{O}_X(1)$ defined on open affines $\operatorname{Spec} A \subseteq Y$ and $\operatorname{Spec} B \subseteq \pi^{-1}(\operatorname{Spec} A)$ by the natural map ${}_AM \otimes_A B \to M$, $m \otimes b \mapsto bm$, where $\mathscr{O}_X(1)\big|_{\operatorname{Spec} B} = \widetilde{M}$.

To define α, pick an irreducible open affine $\operatorname{Spec} A \subseteq Y$ over which $\mathscr{E}$ is free with a basis $\mathscr{B}$. Let $\tau\colon \pi_*\mathscr{O}(1) \to \mathscr{E}$ be the isomorphism from part (a). Then $\Omega_{X/Y}\big|_{\pi^{-1}(\operatorname{Spec} A)}$ is the locally free $\mathscr{O}_{\pi^{-1}(\operatorname{Spec} A)}$-module which for all $y \in \mathscr{B}$ is free over $D_+(y)$ with basis $\{d(x/y) : x \in \mathscr{B} \setminus \{y\}\}$, and α is defined on $D_+(y)$ by

$$\alpha\left(d\left(\frac{x}{y}\right) \otimes y\right) = \tau(x) - \frac{x}{y}\tau(y) \tag{**}$$

for all $x \in \mathscr{B} \setminus \{y\}$.

Both sides of (**) are A-linear in x, so (**) holds for all $x \in \Gamma(\operatorname{Spec} A, \mathscr{E})$. Also, since

$$d\left(\frac{y}{x}\right) \otimes x = \left(\frac{-d(x/y)}{(x/y)^2}\right) = -\frac{y}{x} d\left(\frac{x}{y}\right) \otimes y$$

and

$$\tau(y) - \frac{y}{x}\tau(x) = -\frac{y}{x}\left(\tau(x) - \frac{x}{y}\tau(y)\right)$$

on a suitable nonempty open subset of $\pi^{-1}(\operatorname{Spec} A)$, (**) holds for all nonzero $x, y \in \Gamma(\operatorname{Spec} A, \mathscr{E})$ on some nonempty open subset (depending on x and y). Since $\mathscr{E}$ is torsion free, (**) holds on $D_+(y) \subseteq \pi^{-1}(\operatorname{Spec} A)$ for all such x and y.

Thus the definition of α is independent of the choice of basis of $\mathscr{E}$ over $\operatorname{Spec} A$. In particular the definitions of α are compatible as $\operatorname{Spec} A$ varies, so they glue to give a well-defined natural map $\alpha\colon \Omega_{X/Y}(1) \to \pi^*\mathscr{E}$.

The sequence (*) is therefore exact, since it is exact over open affines in Y by (II, 8.13).

By (II, Ex. 5.16d), we then have

$$\wedge^n \Omega_{X/Y} \cong (\pi^* \wedge^{n+1} \mathscr{E})(-n-1)\ .$$

As for the final assertion, we note by Remark 7.1.1 that for any ring A there is a canonical isomorphism $H^n(\mathbb{P}^n_A, \omega_{\mathbb{P}^n_A/A}) \xrightarrow{\sim} A$ which is invariant under change of basis and functorial in A. By (III, Prop. 8.2) and (III, Prop. 8.5) this gives a canonical isomorphism $R^n\pi_*\omega_{X/Y} \xrightarrow{\sim} \mathscr{O}_Y$ when Y is affine. Since this isomorphism is functorial in A (and in particular is preserved under passing to a principal open affine $\operatorname{Spec} A_f$ in $Y = \operatorname{Spec} A$), these isomorphisms for various open affines in Y glue to give a canonical isomorphism $R^n\pi_*\omega_{X/Y} \xrightarrow{\sim} \mathscr{O}_Y$ for arbitrary Y, as was to be shown.

(c). As noted earlier, this is basically a relative version of (III, 5.1d).

Following the principle mentioned at the beginning of the solution, we start with some global constructions.

Fix $m \in \mathbb{Z}$.

For all open $U \subseteq Y$, tensoring with a section $s \in \Gamma(\pi^{-1}(U), \mathscr{O}_X(m))$ gives a sheaf map $\omega_{X/Y}(-m)\big|_{\pi^{-1}(U)} \to \omega_{X/Y}\big|_{\pi^{-1}(U)}$, which (since $R^n\pi_*$ is a functor) gives a map $(R^n\pi_*(\omega_{X/Y}(-m)))\big|_U \to (R^n\pi_*\omega_{X/Y})\big|_U$. This gives a map

$$\Gamma(U, \pi_*\mathscr{O}_X(m)) \longrightarrow \Gamma(U, \mathscr{H}om_Y(R^n\pi_*(\omega_{X/Y}(-m)), R^n\pi_*\omega_{X/Y}))\ .$$

As U varies, the above maps are compatible with the restriction maps of the relevant sheaves, so this gives a natural map

$$\pi_*\mathscr{O}_X(m) \longrightarrow \mathscr{H}om_Y(R^n\pi_*(\omega_{X/Y}(-m)), R^n\pi_*\omega_{X/Y}) \tag{1}$$

of sheaves on Y. By (III, 8.2) and (III, 8.5), $(R^n\pi_*\omega_{X/Y})\big|_U \cong H^n(\pi^{-1}(U), \omega_{X/Y})^\sim$ and $R^n\pi_*(\omega_{X/Y}(-m))\big|_U \cong H^n(\pi^{-1}(U), \omega_{X/Y}(-m))^\sim$ on open affines $U = \operatorname{Spec} A$ in Y on which $\mathscr{E}$ is trivial. Therefore, by (III, 5.1d), the map (1) is an isomorphism.

By part (b),

$$\begin{aligned}\mathscr{H}om_Y(R^n\pi_*(\omega_{X/Y}(-m)), R^n\pi_*\omega_{X/Y}) &\cong \mathscr{H}om_Y(R^n\pi_*(\omega_{X/Y}(-m)), \mathscr{O}_Y) \\ &\cong (R^n\pi_*(\omega_{X/Y}(-m)))^\vee . \end{aligned} \tag{2}$$

On open affines $U = \operatorname{Spec} A$ as above, we also have that $\pi^{-1}(U) \cong \mathbb{P}^n_A$ and (again by (III 5.1), since $\omega_{X/Y} \cong \mathscr{O}(-n-1)$), that $H^n(\pi^{-1}(U), \omega_{X/Y}(-m))$ is a free A-module of finite rank. Therefore $R^n\pi_*(\omega_{X/Y}(-m))$ is a locally free sheaf on Y of finite rank. We then have

$$\begin{aligned}(\pi_*\mathscr{O}_X(m))^\vee &\cong R^n\pi_*(\omega_{X/Y}(-m)) \\ &\cong R^n\pi_*((\pi^* \wedge^{n+1} \mathscr{E})(-m-n-1)) \\ &\cong (\wedge^{n+1}\mathscr{E}) \otimes R^n\pi_*(\mathscr{O}_X(-m-n-1))\end{aligned}$$

by combining the duals of (1) and (2); by part (b); and by (Ex. 8.3) (respectively). Setting $m = -l-n-1$ and rearranging terms then gives

$$R^n\pi_*(\mathscr{O}_X(l)) \cong (\pi_*\mathscr{O}_X(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee ,$$

as was to be shown.

(d). Following the hint, we first show that $R^i\pi_*(\mathscr{O}_X) = 0$ for all $i > 0$. When $0 < i < n$ this follows from part (a). When $i = n$ this follows from part (c) and the fact that $\pi_*(\mathscr{O}(-n-1)) = 0$ (from part (a)). When $i > n$ this follows from Prop. 8.5 and the fact that if Y is affine then $H^n(X, \mathscr{O}_X)$ can be computed using Čech cohomology with an open cover consisting of $n+1$ elements.

Therefore, Ex. 8.1 applies, which gives

$$\begin{aligned}\chi(\mathscr{O}_X) &= \sum_i (-1)^i h^i(X, \mathscr{O}_X) \\ &= \sum_i (-1)^i h^i(Y, \pi_*\mathscr{O}_X) \\ &= \sum_i (-1)^i h^i(Y, \mathscr{O}_Y) \\ &= \chi(\mathscr{O}_Y) .\end{aligned}$$

Here $h^i(X, \mathscr{O}_X)$ denotes $\dim_k H^i(X, \mathscr{O}_X)$ and similarly for $h^i(Y, \mathscr{O}_Y)$.

Therefore

$$p_a(X) = (-1)^{\dim X}(\chi(\mathscr{O}_X) - 1) = (-1)^{n+\dim Y}(\chi(\mathscr{O}_Y) - 1) = (-1)^n p_a(Y) .$$

Next, for $p_g(X)$, we note that by II Ex. 8.3(b) (applied locally on $\mathbb{A}^n_U$ for open $U \subseteq Y$ over which $\mathscr{E}$ is free), the First Exact Sequence for π is exact on the left also:

$$0 \longrightarrow \pi^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0 .$$

Therefore, by II Ex. 5.16d,

$$\omega_X \cong \omega_{X/Y} \otimes \omega_Y .$$

For any open $U \subseteq Y$ over which $\mathscr{E}$ and ω_Y are trivial, we have

$$\omega_X\big|_{\pi^{-1}(U)} \cong \omega_{X/Y}\big|_{\pi^{-1}(U)} \cong \mathscr{O}_X(-n-1)\big|_{\pi^{-1}(U)} ,$$

and therefore $\Gamma(\pi^{-1}(U), \omega_X) \cong \Gamma(\pi^{-1}(U), \mathscr{O}_X(-n-1)) = 0$ by part (a). Thus

$$p_g(X) = h^0(X, \omega_X) = 0 .$$

(e). From part (d) we get $p_a(X) = -p_a(Y) = -g$ and $p_g(X) = 0$, and its irregularity is $p_g(X) - p_a(X) = g$.

5(NC). (10 points) Give an explicit example showing that (III, Thm. 8.8) is false if X and Y are allowed to be locally noetherian schemes instead of noetherian schemes.

Let k be the field $\mathbb{Q}$, for all $i \in \mathbb{N}$ let X_i be a copy of $\mathbb{P}^1_k$, and let X be the disjoint union $X = X_0 \coprod X_1 \coprod \dots$. Let $\mathscr{F}$ be the sheaf on X whose restriction to X_i is $\mathscr{O}(-i)$ for all i. Then $\mathscr{F}$ is a coherent sheaf on X (see the definition on page 111), because its restriction to X_i is coherent for each i (in more detail, X can be covered by the standard open affines $D_+(x_0)$ and $D_+(x_1)$, where $\mathscr{F}\big|_U \cong \widetilde{k[x]}$ for $U = D_+(x_0)$ and $U = D_+(x_1)$).

Now, letting $g\colon \mathbb{P}^1_k \to \operatorname{Spec} k$ be the canonical morphism, we have that $\mathscr{O}(j)$ has no global sections whenever $j < 0$; therefore $g^*g_*\mathscr{O}(j) = 0$, thus $g^*g_*\mathscr{O}(j) \to \mathscr{O}(j)$ is not surjective. Then, for any $n \in \mathbb{Z}$, choose $i \in \mathbb{N}$ such that $i > n$. Then $\mathscr{F}(n)\big|_{X_i} \cong \mathscr{O}(n-i)$, which implies that $f^*f_*\mathscr{F}(n) \to \mathscr{F}(n)$ is not surjective, because it is not surjective over X_i. Therefore part (a) is false.

Part (b) remains true in this example (and is true in general, because its statement is local on the base).

For part (c), by duality we have $H^1(\mathbb{P}^1_k, \mathscr{O}(j)) \ne 0$ for all $j \le -2$. Therefore, for any given n, we have $R^1f_*\mathscr{F}(n)\big|_{X_i} \cong (H^1(X_i, \mathscr{O}(n-i)))^\sim \ne 0$ for all $i \in \mathbb{N}$ with $i \ge n+2$. This implies that (c) is false.