Math 256B. Solutions to Homework 2

1. (15 points) Hartshorne II Ex. 2.17: *A Criterion for Affineness.*

   (a). Let $f\colon X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

   (b). A scheme X is affine if and only if there is a finite set of elements $f_1, \dots, f_r \in A = \Gamma(X, \mathscr{O}_X)$, such that the open subsets X_{f_i} are affine, and $f_1, \dots, f_r$ generate the unit ideal in A. [*Hint:* Use (Ex. 2.4) and (Ex. 2.16d) above.]

(a). The conditions easily imply that f induces a homeomorphism $\operatorname{sp}(X) \xrightarrow{\sim} \operatorname{sp}(Y)$. For all $P \in Y$ there is some i for which $P \in U_i$; then, since $f\big|_{f^{-1}(U_i)}$ is an isomorphism, $f^\#$ induces an isomorphism of stalks $\mathscr{O}_{Y,P} \to (f_*\mathscr{O}_X)_P$. Since f is a homeomorphism on topological spaces this implies that $f^\#$ is an isomorphism $\mathscr{O}_Y \xrightarrow{\sim} f_*\mathscr{O}_X$, and therefore f is an isomorphism of schemes.

(b). The "$\implies$" direction is trivial (take $r = 1$ and $f_1 = 1$).

For the converse direction, by (II, Ex. 2.4) there is a morphism $f\colon X \to \operatorname{Spec} A$ that induces the identity map $A \to \Gamma(X, \mathscr{O}_X)$. Since $f_1, \dots, f_r$ generate the unit ideal in A, the sets $U_i := D(f_i)$ cover $\operatorname{Spec} A$. We are given that $X_{f_i} = f^{-1}(D(f_i))$ are affine, so

$$X_{f_i} = \operatorname{Spec} \Gamma(X_{f_i}, \mathscr{O}_X) = \operatorname{Spec} A_{f_i}$$

by Ex. 2.16d. (In order to apply Ex. 2.16d, we need to know that X is quasi-compact, and that all sets $X_{f_i} \cap X_{f_j}$ are quasi-compact. The first assertion holds because by assumption X is a finite union of affine, hence quasi-compact, sets. The second assertion holds because by assumption X_{f_i} is affine, so $X_{f_i} \cap X_{f_j} = (X_{f_i})_{f_j}$ is a principal open subset of an affine set, hence affine and therefore quasi-compact.)

Therefore $f^{-1}(U_i) \to U_i$ is an isomorphism for all i, so f is an isomorphism by part (a).

2. (10 points) Hartshorne III Ex. 2.2: Let $X = \mathbb{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{K} \longrightarrow \mathscr{K}/\mathscr{O} \longrightarrow 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathscr{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathscr{O}) = 0$ for all $i > 0$.

   (Note that the first sentence of (II Ex. 1.21) says that we're working with varieties over an algebraically closed field k, as defined in Chapter I, so $X = \mathbb{P}^1$ has no generic point.)

By (II Ex. 1.16a), $\mathscr{K}$ is flasque. We need to show that $\mathscr{K}/\mathscr{O}$ is flasque. By (II Ex. 1.21d), $\mathscr{K}/\mathscr{O}$ is isomorphic to the infinite direct sum $\sum_{P \in X} i_P(I_P)$, where I_P is the sheaf $K/\mathscr{O}_P$ at P and $i_P(I_P)$ is the corresponding skyscraper sheaf.

(It is a general fact of category theory that if a category has finite coproducts and direct limits, then it has infinite coproducts, equal to the direct limit of finite subproducts. So, by (II, Ex. 1.9) and (II, Ex. 1.10), infinite direct sums of sheaves exist.) Also, by (II, Ex. 1.16d), the sheaves $i_P(I_P)$ are flasque. By (II, Ex. 1.9) and (II, Ex. 1.11), global sections of an infinite direct sum of sheaves on a noetherian topological space are given by the direct sums of the global sections of the summands (this can be done for an arbitrary topological space by re-doing (II, Ex. 1.9) for the infinite case). Therefore, since the direct sum of a family of surjections is again a surjection, it follows that the direct sum of a family of flasque sheaves is flasque; in particular, $\mathscr{K}/\mathscr{O}$ is flasque. Thus $0 \to \mathscr{O} \to \mathscr{K} \to \mathscr{K}/\mathscr{O} \to 0$ is a flasque resolution of $\mathscr{O}$.

By Propositions 2.5 and 1.2A, the cohomology of $\mathscr{O}$ is then given by the cohomology of the complex

$$0 \longrightarrow \Gamma(X, \mathscr{K}) \xrightarrow{\alpha} \Gamma(X, \mathscr{K}/\mathscr{O}) \longrightarrow 0 .$$

By (II, Ex. 1.21e), $H^1(X, \mathscr{O}) = \operatorname{coker} \alpha = 0$, and all higher cohomology groups are obviously zero.

3. (10 points) (Vakil 23.2.J,K)

   (a). Let Q be an injective abelian group, and let A be a ring. Show that $\operatorname{Hom}_{\mathbb{Z}}(A, Q)$ is an injective A-module. **Hint:** First describe the A-module structure on $\operatorname{Hom}_{\mathbb{Z}}(A, Q)$. You will only use the fact that $\mathbb{Z}$ is a ring, and that A is an algebra over that ring.

   (b). Show that $\mathfrak{Mod}(A)$ has enough injectives. **Hint:** Let M be an A-module. Find an inclusion $M \hookrightarrow Q$ of abelian groups, such that Q is an injective abelian group. Describe a sequence

$$M \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(A, M) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(A, Q)$$

   of inclusions of A-modules. (The A-module structure on $\operatorname{Hom}_{\mathbb{Z}}(A, M)$ is via the action of A on the left argument A, not on the right argument M.)

(a). Following the hint, we define an A-module structure on $\operatorname{Hom}_{\mathbb{Z}}(A, Q)$ as follows. Let $a \in A$ and $\phi \in \operatorname{Hom}_{\mathbb{Z}}(A, Q)$. Then $a\phi$ is the map $b \mapsto \phi(ab)$; i.e., the action is on the left argument A, not on the right argument Q (as in the hint for (b)).

Now the key to this part is to note that for all A-modules M there is an isomorphism

$$\operatorname{Hom}_A(M, \operatorname{Hom}_{\mathbb{Z}}(A, Q)) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(M, Q) ,$$

given by

$$\psi \in \operatorname{Hom}_A(M, \operatorname{Hom}_{\mathbb{Z}}(A, Q)) \mapsto (m \mapsto \psi(m)(1)) .$$

Indeed, this is clearly additive.

To see that it is injective, let ψ be an element of the kernel. Then $\psi(m)(1) = 0$ for all $m \in M$; therefore $\psi(m)(a) = 0$ for all $a \in A$ and all m because

$$\psi(m)(a) = (a\psi)(m)(1) = \psi(am)(1) = 0 ,$$

and this implies that $\psi(m)=0$ for all m, so $\psi=0$.

To see that is is surjective, let $\phi\in\operatorname{Hom}_{\mathbb{Z}}(M,Q)$. Define $\psi\colon M\to\operatorname{Hom}_{\mathbb{Z}}(A,Q)$ by $m\mapsto(a\mapsto\phi(am))$. This lies in $\operatorname{Hom}_A(M,\operatorname{Hom}_{\mathbb{Z}}(A,Q))$ because

$$\psi(a'm)=(a\mapsto\phi(aa'm))=a'(a\mapsto\phi(am))=a'(\psi(m))\ .$$

Now let $0\longrightarrow M'\longrightarrow M$ be an exact sequence of A-modules. It is also an exact sequence of abelian groups, so the induced map $\operatorname{Hom}_{\mathbb{Z}}(M,Q)\longrightarrow\operatorname{Hom}_{\mathbb{Z}}(M',Q)$ is surjective since Q is an injective abelian group. By the above isomorphism,

$$\operatorname{Hom}_A(M,\operatorname{Hom}_{\mathbb{Z}}(A,Q))\longrightarrow\operatorname{Hom}_A(M',\operatorname{Hom}_{\mathbb{Z}}(A,Q))$$

is also surjective, and this gives that $\operatorname{Hom}_{\mathbb{Z}}(A,Q)$ is an injective A-module.

(b). Following the hint, the second map is an injection because $M\to Q$ is an injection, and it is clear that it is a map of A-modules.

Let the first map be $\phi\colon m\mapsto(a\mapsto am)$. It is an injection because if $\phi(m)=0$ then $\phi(m)(1)=m$ must be zero, and it is an A-module homomorphism because $\phi(a'm)=(a\mapsto aa'm)=a'(a\mapsto am)=a'\phi(m)$.

Thus $\mathfrak{Mod}(A)$ has enough injectives.