Math 256B. Solutions to Homework 3

1. (20 points) Hartshorne III Ex. 2.3: *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X,\mathscr{F})$ denote the group of sections of $\mathscr{F}$ with supports in Y (II, Ex. 1.20).

   (a). Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.
   We denote the right derived functors of $\Gamma_Y(X,\cdot)$ by $H^i_Y(X,\cdot)$. They are the *cohomology groups of* X *with supports in* Y, and coefficients in a given sheaf.

   (b). If $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ is an exact sequence of sheaves with $\mathscr{F}'$ flasque, show that
$$0\to\Gamma_Y(X,\mathscr{F}')\to\Gamma_Y(X,\mathscr{F})\to\Gamma_Y(X,\mathscr{F}'')\to 0$$
   is exact.

   (c). Show that if $\mathscr{F}$ is flasque, then $H^i_Y(X,\mathscr{F})=0$ for all $i>0$.

   (d). If $\mathscr{F}$ is flasque, show that the sequence
$$0\to\Gamma_Y(X,\mathscr{F})\to\Gamma(X,\mathscr{F})\to\Gamma(X-Y,\mathscr{F})\to 0$$
   is exact.

   (e). Let $U=X-Y$. Show that for any $\mathscr{F}$, there is a long exact sequence of cohomology groups
$$\begin{aligned}0&\to H^0_Y(X,\mathscr{F})\to H^0(X,\mathscr{F})\to H^0(U,\mathscr{F}\big|_U)\to\\&\to H^1_Y(X,\mathscr{F})\to H^1(X,\mathscr{F})\to H^1(U,\mathscr{F}\big|_U)\to\\&\to H^2_Y(X,\mathscr{F})\to\dots\ .\end{aligned}$$

   (f). *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathscr{F}$,
$$H^i_Y(X,\mathscr{F})\cong H^i_Y(V,\mathscr{F}\big|_V)\ .$$

(a). Let $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ be an exact sequence of sheaves on X. Then we have a commutative diagram

$$\begin{array}{ccccccc}0&\longrightarrow&\Gamma(X,\mathscr{F}')&\longrightarrow&\Gamma(X,\mathscr{F})&\longrightarrow&\Gamma(X,\mathscr{F}'')\\&&\cup&&\cup&&\cup\\0&\longrightarrow&\Gamma_Y(X,\mathscr{F}')&\xrightarrow{\alpha}&\Gamma_Y(X,\mathscr{F})&\xrightarrow{\beta}&\Gamma_Y(X,\mathscr{F}'')\end{array}$$

in which the top row is exact. Immediately this gives that α is injective and that $\beta\circ\alpha=0$, so $\ker\beta\supseteq\operatorname{im}\alpha$. Regarding $\mathscr{F}'$ as a subsheaf of $\mathscr{F}$, we have

$$\Gamma_Y(X,\mathscr{F}')=\Gamma(X,\mathscr{F}')\cap\Gamma_Y(X,\mathscr{F})\ ,$$

so any $s\in\ker\beta$ lies in $\Gamma(X,\mathscr{F}')$ and in $\Gamma_Y(X,\mathscr{F})$, hence in $\Gamma_Y(X,\mathscr{F}')$. Therefore the sequence is exact.

(b). By part (a), we only need to show that the map $\Gamma_Y(X,\mathscr{F})\to\Gamma_Y(X,\mathscr{F}'')$ is surjective. Let $s''\in\Gamma_Y(X,\mathscr{F}'')$. By (II Ex. 1.16b), there is a section $s\in\Gamma(X,\mathscr{F})$ mapping to $s''\in\Gamma(X,\mathscr{F}'')$. Now $s''\big|_{X\setminus Y}=0$, so $s\big|_{X\setminus Y}$ lies in $\mathscr{F}'$. Since $\mathscr{F}'$ is flasque, there is a $t\in\mathscr{F}'(X)$ such that $t\big|_{X\setminus Y}=s\big|_{X\setminus Y}$. Noting that t is taken to 0 in $\mathscr{F}''$, we have that $s-t$ also maps to s'' in $\mathscr{F}''$. Also, $(s-t)\big|_{X\setminus Y}=0$, so $s-t\in\Gamma_Y(X,\mathscr{F})$. Thus β is surjective, as was to be shown.

(c). The proof of Proposition 2.5 carries over almost verbatim, using part (b) in place of (II Ex. 1.16b).

(d). The first map is injective by definition, and the second is surjective by flasqueness. If $s\in\Gamma_Y(X,\mathscr{F})$, then $s\big|_{X\setminus Y}=0$ since all stalks are 0, so the kernel of the second map contains the image of the first map. Conversely, if s is in the kernel of the second map, then $s\big|_{X\setminus Y}=0$, so $s\in\Gamma_Y(X,\mathscr{F})$.

Or: Since $\mathscr{F}$ is flasque, the sequence $0\longrightarrow\mathscr{H}^0_Y(\mathscr{F})\longrightarrow\mathscr{F}\longrightarrow j_*(\mathscr{F}\big|_U)\longrightarrow 0$ is exact by (II, Ex. 1.20), where $U=X\setminus Y$ and $j\colon U\to X$ is the inclusion map. Taking global sections gives that $0\longrightarrow\Gamma_Y(X,\mathscr{F})\longrightarrow\Gamma(X,\mathscr{F})\longrightarrow\Gamma(X\setminus Y,\mathscr{F})\longrightarrow 0$ is exact, except possibly at $\Gamma(X\setminus Y,\mathscr{F})$. But, since $\mathscr{F}$ is flasque, $\Gamma(X,\mathscr{F})\to\Gamma(X\setminus Y,\mathscr{F})$ is surjective, and we are done.

(e). Let $0\longrightarrow\mathscr{F}\longrightarrow\mathscr{I}^{\cdot}$ be a flasque resolution of $\mathscr{F}$, and consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_Y(X,\mathscr{I}^0) & \longrightarrow & \Gamma(X,\mathscr{I}^0) & \longrightarrow & \Gamma(U,\mathscr{I}^0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_Y(X,\mathscr{I}^1) & \longrightarrow & \Gamma(X,\mathscr{I}^1) & \longrightarrow & \Gamma(U,\mathscr{I}^1) & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

By part (d), the rows are exact. The columns are complexes. By part (c) and Prop. 1.2A, the cohomology of the first two columns give $H^*_Y(X,\mathscr{F})$ and $H^*(X,\mathscr{F})$, respectively. Since flasqueness of sheaves and exactness of sequences of sheaves is preserved under restricting to an open subset, the cohomology of the third column is $H^*(U, \mathscr{F}\big|_U)$. The desired long exact sequence then follows by applying the Snake Lemma.

(f). First, we claim that the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(V, \mathscr{F}\big|_V)$ induces an isomorphism $\alpha\colon \Gamma_Y(X, \mathscr{F}) \xrightarrow{\sim} \Gamma_Y(V, \mathscr{F}) = \Gamma_Y(V, \mathscr{F}\big|_V)$. First of all, it is clear that if $s \in \Gamma_Y(X, \mathscr{F})$, then $s\big|_V \in \Gamma_Y(V, \mathscr{F})$. Secondly, if $s \in \ker\alpha$, then $s\big|_V = 0$; also, $s\big|_{X\setminus Y} = 0$ since $s \in \Gamma_Y$; therefore $s = 0$ and thus α is injective. Finally, let $t \in \Gamma_Y(V, \mathscr{F})$. Since $t\big|_{V\setminus Y} = 0$, we can glue t with the section $0 \in \Gamma(X \setminus Y, \mathscr{F})$ to give a section $s \in \Gamma(X, \mathscr{F})$. Clearly $s\big|_{X\setminus Y} = 0$, so $s \in \Gamma_Y(X, \mathscr{F})$; moreover $\alpha(s) = t$ by construction, so α is surjective. This proves the claim.

Now let $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^{\cdot}$ be a flasque resolution of $\mathscr{F}$, so, by part (c) and Prop. 1.2A,

$$H^*_Y(X, \mathscr{F}) = h^*(\Gamma_Y(X, \mathscr{I}^{\cdot}))\ .$$

But also $0 \longrightarrow \mathscr{F}\big|_V \longrightarrow \mathscr{I}^{\cdot}\big|_V$ is a flasque resolution of $\mathscr{F}\big|_V$, so

$$H^*_Y(V, \mathscr{F}\big|_V) = h^*(\Gamma_Y(V, \mathscr{I}^{\cdot}\big|_V))\ .$$

By the claim, we then have

$$H^*_Y(X, \mathscr{F}) = h^*(\Gamma_Y(X, \mathscr{I}^{\cdot})) \cong h^*(\Gamma_Y(V, \mathscr{I}^{\cdot}\big|_V)) = H^*_Y(V, \mathscr{F}\big|_V)\ .$$

Moreover, the isomorphism comes from restriction maps, so it is natural and functorial.

2. (10 points) Hartshorne III Ex. 3.1: Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine. [*Hint:* Use (3.7), and for any coherent sheaf $\mathscr{F}$ on X, consider the filtration $\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \dots$, where $\mathscr{N}$ is the sheaf of nilpotent elements on X.]

First note that X_{red} is a closed subscheme of X; hence if X is affine then so is X_{red} (by II Ex. 3.11b or the proof of II 5.9).

To show the converse, let $\mathscr{F}$ be a coherent sheaf on X, let $\mathscr{N}$ be the sheaf of nilpotent elements on X, and for all $i \in \mathbb{N}$ let $\mathscr{F}_i = \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}$. Then each $\mathscr{F}_i$ comes from a sheaf on X_{red}; hence, by (3.7), $H^p(X_{\text{red}}, \mathscr{F}_i) = 0$ for all $p > 0$. By Lemma 2.10, we therefore have $H^p(X, \mathscr{F}_i) = 0$ for all $p > 0$.

For sufficiently large m we have $\mathscr{N}^{m+1} = 0$ (by an argument using quasi-compactness and finite generation of the ideal of nilpotents in a noetherian ring), hence $\mathscr{N}^{m+1}\mathscr{F} = 0$, so $H^p(X, \mathscr{N}^{m+1}\mathscr{F}) = 0$ for all $p > 0$. By considering the long exact sequence in cohomology attached to the short exact sequence

$$0 \longrightarrow \mathscr{N}^{m+1}\mathscr{F} \longrightarrow \mathscr{N}^m\mathscr{F} \longrightarrow \mathscr{F}_m \longrightarrow 0\ ,$$

it follows that $H^p(X, \mathscr{N}^m\mathscr{F}) = 0$ for all $p > 0$; by descending induction and a similar argument, we obtain that $H^p(X, \mathscr{N}^i\mathscr{F}) = 0$ for all $p > 0$ and all $0 \le i \le m$. Thus $H^p(X, \mathscr{F}) = 0$ for all $p > 0$, so by (3.7) it follows that X is affine.

3(NC). (10 points) Hartshorne III Ex. 3.2: Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

In one direction it's easy: If X is affine, then all of its irreducible components are also affine, since they are closed subschemes (II Ex. 3.11b).

To show the converse, we start with two lemmas.

Lemma. *Let X be a noetherian scheme, let Z be a closed subscheme of X with associated ideal sheaf $\mathscr{I}$, and suppose that Z is affine. Let $\mathscr{F}$ be a quasi-coherent sheaf on X that is killed by $\mathscr{I}$. Then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. First we claim that there is a quasi-coherent sheaf $\mathscr{G}$ on Z such that $\mathscr{F} \cong i_*\mathscr{G}$, where $i\colon Z \to X$ is the inclusion map. Indeed, in the affine case $X = \operatorname{Spec} A$, we have $\mathscr{I} \cong \widetilde{I}$ and $\mathscr{F} \cong \widetilde{M}$, where I is an ideal in A and M is an A-module. Moreover, $Y = \operatorname{Spec}(A/I)$ and $IM = 0$. Therefore M has a well-defined structure as a module over A/I, and we can let $\mathscr{G} = \widetilde{M}$ using this structure. Then $\mathscr{F} \cong i_*\mathscr{G}$ by (II, Prop. 5.2d).

The above construction commutes with localization consisting of passing from $\operatorname{Spec} A$ to $\operatorname{Spec} A_f$ for any $f \in A$; therefore the sheaves $\widetilde{M}$ as above glue to give a well-defined quasi-coherent sheaf $\mathscr{G}$ on Z such that $\mathscr{F} \cong i_*\mathscr{G}$ (as above), by Vakil Thm. 13.3.2c.

Since Z is noetherian, Lemma 2.10 and Theorem 3.5 then give

$$H^i(X, \mathscr{F}) = H^i(Z, \mathscr{G}) = 0 \quad \text{for all } i > 0 \ . \qquad \square$$

Lemma. *Let X be a reduced noetherian scheme. Let X_1 and X_2 be reduced closed subschemes of X such that $X = X_1 \cup X_2$ (topologically). If X_1 and X_2 are affine, then so is X.*

Proof. Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be the sheaves of ideals of X_1 and X_2, respectively. Then $\mathscr{I}_1\mathscr{I}_2 \subseteq \mathscr{I}_1 \cap \mathscr{I}_2 = 0$, since $\mathscr{I}_i \cap \mathscr{I}_2$ is associated to a closed subscheme of X whose underlying topological space is all of X, and since X is reduced.

Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Consider the exact sequence

$$0 \longrightarrow \mathscr{I}_1\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{I}_1\mathscr{F} \longrightarrow 0 \ .$$

Since $\mathscr{I}_1\mathscr{F}$ and $\mathscr{F}/\mathscr{I}_1\mathscr{F}$ are killed by $\mathscr{I}_2$ and $\mathscr{I}_1$, respectively, they are acyclic by the previous lemma. The long exact sequence in cohomology then implies that $\mathscr{F}$ is acyclic, so X is affine by Theorem 3.7. $\square$

Now let X be a reduced noetherian scheme, and let $X_1, \dots, X_n$ be its irreducible components (with reduced induced subscheme structure). We want to show that if $X_1, \dots, X_n$ are all affine, then X is affine. If $n \le 1$ then this is trivial. For larger values of n this follows from the above lemma by induction.