Math 256B. Solutions to Homework 4

1(NC). (10 points) Hartshorne III Ex. 4.1: Let $f\colon X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \ge 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}) \ .$$

[*Hint:* Use (II, 5.8).]

Let $\mathscr{V} = (V_i)_{i\in I}$ be an open affine cover of Y, and let $\mathscr{U} = (U_i)_{i\in I}$ be the corresponding open cover of X, with $U_i = f^{-1}(V_i)$ for all i. Since f is an affine morphism, $\mathscr{U}$ is an open *affine* cover of X (II Ex. 5.17a). Since X is noetherian, $f_*\mathscr{F}$ is quasi-coherent by (II 5.8c). Therefore, by Theorem 4.5 it will suffice to construct natural isomorphisms

$$\check{H}^p(\mathscr{U}, \mathscr{F}) \cong \check{H}^p(\mathscr{V}, f_*\mathscr{F})$$

for all $p \in \mathbb{N}$.

But now

$$C^p(\mathscr{V}, f_*\mathscr{F}) = \prod_{i_0<\dots<i_p} (f_*\mathscr{F})(V_{i_0\dots i_p}) = \prod_{i_0<\dots<i_p} \mathscr{F}(U_{i_0\dots i_p}) = C^p(\mathscr{U}, \mathscr{F})$$

(where the middle equality holds by definition of the direct image sheaf), and the coboundary maps obviously correspond since restrictions on $\mathscr{F}$ and $f_*\mathscr{F}$ are compatible. Thus

$$\check{H}^p(\mathscr{U}, \mathscr{F}) \cong \check{H}^p(\mathscr{V}, f_*\mathscr{F})$$

for all $p \in \mathbb{N}$.

These maps are natural because the isomorphism $C^\cdot(\mathscr{U}, \mathscr{F}) \cong C^\cdot(\mathscr{V}, f_*\mathscr{F})$ is natural.

2. (10 points) Hartshorne III Ex. 4.3: Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, and let $U = X - \{(0,0)\}$. Using a suitable cover of U by open affines, show that $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j : i, j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—see (I, Ex. 3.6).)

Let $\mathscr{U} = \{U_0, U_1\}$, where

$$U_0 = D(x) = \operatorname{Spec} k[x,y]_x \qquad \text{and} \qquad U_1 = D(y) = \operatorname{Spec} k[x,y]_y \ .$$

This is an open affine cover of U, so by Theorem 4.5, $H^1(U, \mathscr{O}_U) \cong \check{H}^1(\mathscr{U}, \mathscr{O}_U)$. Computing Čech cohomology, we have

$$C^0(\mathscr{U}, \mathscr{O}_U) = \mathscr{O}_U(U_0) \times \mathscr{O}_U(U_1) = k[x,y]_x \times k[x,y]_y$$

and

$$C^1(\mathscr{U}, \mathscr{O}_U) = \mathscr{O}_U(U_0 \cap U_1) = k[x,y]_{xy}$$

with the map $d^0\colon C^0(\mathscr{U}, \mathscr{O}_U) \to C^1(\mathscr{U}, \mathscr{O}_U)$ given by $(f,g) \mapsto f - g$. We then have $\check{H}^1(\mathscr{U}, \mathscr{O}_U) \cong \operatorname{coker} d^0$. As vector spaces over k, we have

$$k[x,y]_{xy} = \bigoplus_{i,j\in\mathbb{Z}} k \cdot x^i y^j$$

and

$$\operatorname{im} d^0 = \bigoplus_{\substack{i,j\in\mathbb{Z} \\ j\ge 0}} k \cdot x^i y^j + \bigoplus_{\substack{i,j\in\mathbb{Z} \\ i\ge 0}} k \cdot x^i y^j \ ,$$

so the cokernel has basis

$$\{x^i y^j : i,j \in \mathbb{Z} \text{ and } (i,j) \notin (\mathbb{Z}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{Z})\} = \{x^i y^j : i,j < 0\} \ ,$$

as was to be shown. (Note that this implies that U is not affine, since if U was affine it would contradict Theorem 3.5. This is in addition to the reason given in the problem statement.)

3. (15 points) Hartshorne III Ex. 4.5: For any ringed space $(X, \mathscr{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves (II, § 6). Show that $\operatorname{Pic} X \cong H^1(X, \mathscr{O}_X^*)$, where $\mathscr{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathscr{O}_X)$, with multiplication as the group operation. [*Hint:* For any invertible sheaf $\mathscr{L}$ on X, cover X by open sets U_i on which $\mathscr{L}$ is free, and fix isomorphisms $\phi_i\colon \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}\big|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\phi_i^{-1} \circ \phi_j$ of $\mathscr{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U}, \mathscr{O}_X^*)$. Now use (Ex. 4.4).]

First, fix an open cover $\mathfrak{U} = (U_i)_{i\in I}$ and let $G_{\mathfrak{U}}$ be the set of pairs $(\mathscr{L}, (\phi_i)_{i\in I})$ with $\mathscr{L} \in \operatorname{Pic} X$ and $\phi_i\colon \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}\big|_{U_i}$ an isomorphism for all i. Define a binary operation $* : G_{\mathfrak{U}} \times G_{\mathfrak{U}} \to G_{\mathfrak{U}}$ by

$$(\mathscr{L}, (\phi_i)) * (\mathscr{M}, (\psi_i)) = (\mathscr{L} \otimes \mathscr{M}, (\phi_i \star \psi_i)) \ ,$$

where $\phi_i \star \psi_i$ is the composition

$$\phi_i \star \psi_i \colon \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i} \otimes \mathscr{O}_{U_i} \xrightarrow{\phi_i \otimes \psi_i} (\mathscr{L} \otimes \mathscr{M})\big|_{U_i} \ .$$

With this operation, $G_{\mathfrak{U}}$ becomes a group, with identity element $(\mathscr{O}_X, (\phi_i))$, where ϕ_i is the identity map on $\mathscr{O}_{U_i}$. Also, $(\mathscr{L}, (\phi_i)) \mapsto \mathscr{L}$ defines a group homomorphism

$$\Phi_{\mathfrak{U}} \colon G_{\mathfrak{U}} \to \operatorname{Pic} X \ .$$

The kernel of this map is obviously the set of pairs $(\mathscr{O}_X, (\phi_i))$ with $\phi_i \in \operatorname{Aut} \mathscr{O}_{U_i}$ for all i. Here and in what follows, "Aut" always refers to automorphism as sheaves of $\mathscr{O}_X$-modules (or $\mathscr{O}_{U_i}$-modules, as appropriate). The image of $\Phi_{\mathfrak{U}}$ is the set of all line sheaves $\mathscr{L}$ whose restrictions to U_i are trivial for all i.

Lemma. *Let* $(X, \mathscr{O}_X)$ *be a ringed space. Then the map* $\rho_X\colon \operatorname{Aut}\mathscr{O}_X \longrightarrow \Gamma(X, \mathscr{O}_X)^*$ *given by* $\phi \mapsto \phi(1)$ *is a group isomorphism.*

Proof. An $\mathscr{O}_X$-module homomorphism $\mathscr{O}_X \to \mathscr{O}_X$ is determined by its action on a generator, i.e., 1, so ρ_X is injective. It is also obviously a group homomorphism, since $\phi \circ \psi$ takes 1 to $\phi(\psi(1)) = \phi(\psi(1)\cdot 1) = \psi(1)\cdot\phi(1)$. Finally, given $u \in \Gamma(X, \mathscr{O}_X)^*$, the map $\phi\colon \mathscr{O}_X \to \mathscr{O}_X$ given by $s \mapsto u\big|_U \cdot s$ for sections $s \in \mathscr{O}_X(U)$ is an $\mathscr{O}_X$-module homomorphism with inverse given by $s \mapsto u^{-1}\big|_U \cdot s$, so it lies in $\operatorname{Aut}\mathscr{O}_X$ and therefore ρ_X is surjective. □

Define a function

$$\Theta_{\mathfrak{U}}\colon G_{\mathfrak{U}} \longrightarrow C^1(\mathfrak{U}, \mathscr{O}_X^*)$$

by

$$(\Theta_{\mathfrak{U}}(\mathscr{L}, (\phi_i)))_{\alpha\beta} = \rho_{U_{\alpha\beta}}\big(\phi_\alpha^{-1}\big|_{U_{\alpha\beta}} \circ \phi_\beta\big|_{U_{\alpha\beta}}\big) \in \mathscr{O}_X(U_{\alpha\beta})^* = \mathscr{O}_X^*(U_{\alpha\beta})$$

for all $\alpha < \beta$ in I. This is clearly a group homomorphism.

If $(\mathscr{L}, (\phi_i)) \in G_{\mathfrak{U}}$ and $\alpha < \beta < \gamma$ in I, then the "cocycle condition"

$$\phi_\alpha^{-1} \circ \phi_\gamma = (\phi_\alpha^{-1} \circ \phi_\beta) \circ (\phi_\beta^{-1} \circ \phi_\gamma)$$

holds on $U_{\alpha\beta\gamma}$, so

$$\Theta_{\mathfrak{U}}(\mathscr{L}, (\phi_i))_{\alpha\gamma} = \Theta_{\mathfrak{U}}(\mathscr{L}, (\phi_i))_{\alpha\beta} \cdot \Theta_{\mathfrak{U}}(\mathscr{L}, (\phi_i))_{\beta\gamma}$$

holds on $U_{\alpha\beta\gamma}$ and therefore $d^1(\Theta_{\mathfrak{U}}(\mathscr{L}, (\phi_i))) = 0$; i.e., $\Theta_{\mathfrak{U}}(\mathscr{L}, (\phi_i))$ is a cocycle in $C^1(\mathfrak{U}, \mathscr{O}_X^*)$. Conversely, given a cocycle $(\sigma_{\alpha\beta}) \in C^1(\mathfrak{U}, \mathscr{O}_X^*)$, the automorphisms $\sigma_{\alpha\beta} \in \operatorname{Aut}\mathscr{O}_{U_{\alpha\beta}}$ satisfy the condition $\sigma_{\alpha\gamma} = \sigma_{\beta\gamma} \circ \sigma_{\alpha\beta}$ in $U_{\alpha\beta\gamma}$, so by (II Ex. 1.22) they glue to give a line sheaf $\mathscr{L}$ for which a trivialization in $G_{\mathfrak{U}}$ maps to $(\sigma_{\alpha\beta})$. Therefore $\Theta_{\mathfrak{U}}$ maps onto the set of cocycles. Thus it defines a surjective map

$$\Psi_{\mathfrak{U}}\colon G_{\mathfrak{U}} \longrightarrow \check{H}^1(\mathfrak{U}, \mathscr{O}_X^*)\ .$$

We now claim that $\ker\Phi_{\mathfrak{U}} = \ker\Psi_{\mathfrak{U}}$. Indeed, first let $(\mathscr{L}, (\phi_i)) \in \ker\Psi_{\mathfrak{U}}$. Then $\Theta(\mathscr{L}, (\phi_i))$ is a coboundary in $C^1(\mathfrak{U}, \mathscr{O}_X^*)$, so there is an element

$$u = (u_\alpha)_{\alpha\in I} \in C^0(\mathfrak{U}, \mathscr{O}_X^*)$$

such that $\phi_\alpha^{-1} \circ \phi_\beta = u_\alpha^{-1} \cdot u_\beta$ in $U_{\alpha\beta}$ for all $\alpha < \beta$ in I. Therefore $u_\beta^{-1}\phi_\beta = u_\alpha^{-1}\phi_\alpha$ for all $\alpha < \beta$, so these maps glue to give an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{L}$. Thus $(\mathscr{L}, (\phi_i))$ lies in $\ker\Phi_{\mathfrak{U}}$.

Conversely, if $(\mathscr{L}, (\phi_i)) \in \ker\Phi_{\mathfrak{U}}$ then there is an isomorphism $\psi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{O}_X$. Then, for all i, $\psi\big|_{U_i} \circ \phi_i$ lies in $\operatorname{Aut}\mathscr{O}_{U_i}$, and therefore corresponds to $u_i \in \mathscr{O}_X^*(U_i)$. Moreover, for all $\alpha < \beta$ in I, $\phi_\alpha^{-1} \circ \phi_\beta = (\psi\big|_{U_\alpha} \circ \phi_\alpha)^{-1} \circ (\psi\big|_{U_\beta} \circ \phi_\beta)$ is multiplication by $u_\alpha^{-1}u_\beta$ on $U_{\alpha\beta}$. Thus $\Theta(\mathscr{L},(\phi_i))$ is a coboundary, and therefore $(\mathscr{L},(\phi_i))$ lies in $\ker\Psi_{\mathfrak{U}}$.

Since the kernels of $\Phi_{\mathfrak{U}}$ and $\Psi_{\mathfrak{U}}$ coincide, they define an isomorphism

$$\check{H}^1(\mathfrak{U},\mathscr{O}_X^*) \xrightarrow{\sim} \operatorname{im}\Phi_{\mathfrak{U}} .$$

Now let $\mathfrak{V}=(V_j)_{j\in J}$ be a refinement of $\mathfrak{U}$ and let $\lambda\colon J\to I$ be a function satisfying $V_j\subseteq U_{\lambda(j)}$ for all $j\in J$. Define a map $G_{\mathfrak{U}}\to G_{\mathfrak{V}}$ by $(\mathscr{L},(\phi_i))\mapsto(\mathscr{L},(\psi_j))$, where $\psi_j=\phi_{\lambda(j)}\big|_{V_j}$ for all $j\in J$. We then have a diagram

$$\begin{array}{ccccc}
\check{H}^1(\mathfrak{U},\mathscr{O}_X^*) & \longleftarrow & G_{\mathfrak{U}} & & \\
\downarrow & & \downarrow & \searrow & \\
 & & & & \operatorname{Pic} X \\
 & & & \nearrow & \\
\check{H}^1(\mathfrak{V},\mathscr{O}_X^*) & \longleftarrow & G_{\mathfrak{V}} & &
\end{array} .$$

The triangle on the right obviously commutes, and the square on the left commutes because the definition of Θ is compatible with restriction. This gives an isomorphism

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U},\mathscr{O}_X^*) \xrightarrow{\sim} \varinjlim_{\mathfrak{U}} \operatorname{im}\Phi_{\mathfrak{U}} .$$

Since the groups $\operatorname{im}\Phi_{\mathfrak{U}}$ are all subgroups of $\operatorname{Pic} X$ and since the maps in the directed system are all inclusion maps, this direct limit is really a union. Moreover, this union is all of $\operatorname{Pic} X$ since by definition a line sheaf has a trivializing open cover. Thus, by Ex. 4.4, we have an isomorphism

$$H^1(X,\mathscr{O}_X^*) \xrightarrow{\sim} \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U},\mathscr{O}_X^*) \xrightarrow{\sim} \operatorname{Pic} X ,$$

as was to be shown.