Math 256B. Solutions to Homework 5

1. (15 points) Hartshorne III Ex. 4.7: Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous equation $f(x_0,x_1,x_2)=0$ of degree d. (Do not assume f is irreducible.) Assume that $[1:0:0]$ is not on X. Then show that X can be covered by the two affine subsets $U=X\cap\{x_1\ne 0\}$ and $V=X\cap\{x_2\ne 0\}$. Now calculate the Čech complex

$$\Gamma(U,\mathscr{O}_X)\oplus\Gamma(V,\mathscr{O}_X)\longrightarrow\Gamma(U\cap V,\mathscr{O}_X)$$

explicitly, and thus show that

$$\begin{aligned}\dim H^0(X,\mathscr{O}_X)&=1\ ,\\ \dim H^1(X,\mathscr{O}_X)&=\frac{1}{2}(d-1)(d-2)\ .\end{aligned}$$

We have $D_+(x_1)\cup D_+(x_2)=\mathbb{P}^2_k\setminus\{[1:0:0]\}$ and $[1:0:0]\notin X$; therefore U and V cover X. Then

$$\begin{aligned}U&=\operatorname{Spec} k[u,v]/(f(u,1,v))\ ,\\ V&=\operatorname{Spec} k[s,t]/(f(s,t,1))\ ,\qquad\text{and}\\ U\cap V&=\operatorname{Spec} k[u,v,v^{-1}]/(f(u,1,v))\end{aligned}$$

with $u=x_0/x_1$, $v=x_2/x_1$, $s=x_0/x_2$, and $t=x_1/x_2$.

The condition $[1:0:0]\notin X$ also implies that the coefficient of x_0^d in f is nonzero. By modding out by $f(u,1,v)$ or $f(s,t,1)$ as a polynomial in u or s, respectively, it follows that the affine rings $\Gamma(U,\mathscr{O}_X)$, $\Gamma(V,\mathscr{O}_X)$, and $\Gamma(U\cap V,\mathscr{O}_X)$ of U, V, and $U\cap V$, respectively, have bases

$$\begin{aligned}B_U&:=\{u^iv^j:0\le i<d\text{ and }j\ge 0\}\\ B_V&:=\{s^\ell t^m:0\le \ell<d\text{ and }m\ge 0\}\qquad\text{and}\\ B_{U\cap V}&:=\{u^iv^j:0\le i<d\}\ ,\end{aligned}$$

respectively, as vector spaces over k. These bases give rise to bases for the components of the Čech complex. Since the restriction maps take elements of B_U and B_V to elements of $B_{U\cap V}$, and since these restriction maps make up the coboundary map of the Čech complex, computing Čech cohomology with respect to the open cover $\{U,V\}$ reduces to looking at the images of B_U and B_V in $B_{U\cap V}$. These images are, respectively,

$$\{u^iv^j\in B_{U\cap V}:j\ge 0\}\qquad\text{and}\qquad\{u^iv^j\in B_{U\cap V}:i+j\le 0\}\ .\tag{*}$$

Here we are using the fact that the map $B_V\to B_{U\cap V}$ is given by $s^\ell t^m\mapsto u^\ell v^{-\ell-m}$.

(Note that the images of $f(u,1,v)$ and $f(s,t,1)$ in $k[u,v,v^{-1}]$ are not the same, but they differ by the factor x_1^n/x_2^n, which is a unit, so the results of modding out are compatible.)

The cohomology group $\check{H}^0(\{U,V\},\mathscr{O}_X)$ is determined by the intersection of the two images (*). The conditions $i\ge 0$, $j\ge 0$, $i+j\le 0$ determine this intersection, which therefore consists only of one element, corresponding to $i=j=0$. Thus

$$\dim \check{H}^0(\{U,V\},\mathscr{O}_X) = 1 \; .$$

The first Čech cohomology group is determined by the cokernel of the coboundary map

$$\Gamma(U,\mathscr{O}_X)\times\Gamma(V,\mathscr{O}_X)\longrightarrow\Gamma(U\cap V,\mathscr{O}_X) \; ,$$

and therefore by the complement of the union of the two images (*). This complement is

$$\{u^iv^j : 0\le i<d\,,\ j<0\,, \text{ and } i+j>0\,\} = \{u^iv^j : 0\le i<d \text{ and } -i<j<0\,\} \; .$$

The number of elements of this set is $0+1+2+\dots+(d-2)=(d-1)(d-2)/2$, so

$$\dim \check{H}^1(\{U,V\},\mathscr{O}_X) = \frac{(d-1)(d-2)}{2} \; .$$

These dimensions also give the dimensions of the corresponding derived-functor cohomology groups, by Theorem 4.5.

2. (10 points) Hartshorne III Ex. 5.1: Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum (-1)^i \dim_k H^i(X,\mathscr{F}) \; .$$

If

$$0\longrightarrow\mathscr{F}'\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}''\longrightarrow 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F})=\chi(\mathscr{F}')+\chi(\mathscr{F}'')$.

We start with a general lemma.

Lemma. *Let $\mathscr{C}$ be an abelian category and let $\delta\colon \operatorname{Ob}\mathscr{C}\to\mathbb{Z}$ be a function such that $\delta(X)=\delta(X')+\delta(X'')$ for all short exact sequences*

$$0\longrightarrow X'\longrightarrow X\longrightarrow X''\longrightarrow 0$$

in $\mathscr{C}$. Then

$$\sum_{i=0}^{n}(-1)^i\delta(X_i) = 0 \qquad (*)$$

for all exact sequences

$$0 \longrightarrow X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} X_n \longrightarrow 0$$

in $\mathscr{C}$ *.*

Proof. Use induction on n. If $n \le 2$ then this follows from the assumption on δ (adding zeroes to the sequence if necessary).

So suppose $n \ge 3$. Then we have exact sequences

$$0 \longrightarrow \ker d_2 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \dots \longrightarrow X_n \longrightarrow 0 \tag{1}$$

and

$$0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \operatorname{im} d_1 \longrightarrow 0 , \tag{2}$$

implying that

$$\sum_{i=2}^{n} (-1)^i \delta(X_i) = \delta(\ker d_2) = \delta(\operatorname{im} d_1) = \delta(X_1) - \delta(X_0)$$

by the inductive hypothesis applied to (1), by exactness at X_2, and by the assumption on δ applied to (2). This gives (*). □

Now consider the problem at hand. If $\mathscr{F}$ is a coherent sheaf on X, then $H^i(X, \mathscr{F})$ is finite dimensional (over k) for all i by Theorem 5.2a, and there are only finitely many nonzero terms in the sum defining $\chi(\mathscr{F})$ since X is finite dimensional, by Theorem 2.7. Thus $\chi(\mathscr{F})$ is defined for all coherent sheaves $\mathscr{F}$ on X.

Also, for all short exact sequences as in the problem, the corresponding long exact sequence in cohomology has only finitely many nonzero terms, and each term is a finite-dimensional vector space over k. Applying the lemma to this sequence, with $\mathscr{C}$ equal to the category of finite-dimensional vector spaces over k and $\delta(X) = \dim_k X$ gives

$$\begin{aligned} 0 &= h^0(X, \mathscr{F}') - h^0(X, \mathscr{F}) + h^0(\mathscr{F}'') - h^1(X, \mathscr{F}') + h^1(X, \mathscr{F}) - h^1(X, \mathscr{F}'') + \dots \\ &= \sum_{i \ge 0} (-1)^i h^i(X, \mathscr{F}') - \sum_{i \ge 0} (-1)^i h^i(X, \mathscr{F}) + \sum_{i \ge 0} (-1)^i h^i(X, \mathscr{F}'') \\ &= \chi(\mathscr{F}') - \chi(\mathscr{F}) + \chi(\mathscr{F}'') , \end{aligned}$$

as was to be shown. Here $h^i(X, \mathscr{F})$ denotes $\dim_k H^i(X, \mathscr{F})$ for all i.