Math 256B. Solutions to Homework 6

1. (10 points) [This exercise generalizes (II Ex. 5.9b.)] Let A be a noetherian ring, let S be a graded ring, finitely generated by S_1 over S_0, and assume that $S_0 = A$. Let M be a finitely-generated graded S-module. By (II, Ex. 5.9a), there is a natural map $\alpha\colon M \to \Gamma_*(\widetilde{M})$. Let $X = \operatorname{Proj} S$.

   Show that the map α is an isomorphism in all large enough degrees; i.e.,

$$\alpha_d\colon M_d \to \Gamma(X, \widetilde{M}(d))$$

is an isomorphism for all $d \gg 0$. Use cohomology. [*Hint*: Use methods from the proof of (III, 5.2).]

Let $m_1, \dots, m_r$ be a finite homogeneous generating set for M, and for all i let q_i be minus the degree of m_i. Then we have a graded surjection $\bigoplus S(q_i) \to M$ (preserving degrees). Let K be the kernel of this map; then

$$0 \to K \to \bigoplus S(q_i) \to M \to 0$$

is an exact sequence of graded S-modules. Since S is noetherian, K is finitely generated, so by Theorem 5.2 $H^1(X, \widetilde{K}(n)) = 0$ for all sufficiently large n, which we now assume. We then have a commutative diagram

$$\begin{array}{ccccc}
\bigoplus S_{q_i+n} & \longrightarrow & M_n & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \alpha'_n} & & \big\downarrow{\scriptstyle \alpha_n} & & \\
\Gamma(X, \bigoplus \mathscr{O}(q_i+n)) & \longrightarrow & \Gamma(X, \widetilde{M}(n)) & \longrightarrow & 0
\end{array}$$

with exact rows. By the "Major Corollary" of Thm. 5.2 (proved in class on February 28), α'_n is surjective, so α_n is also surjective by an easy arrow chase.

It remains to show that α_n is injective for all sufficiently large n. Let K be the kernel of α; we then need to show that $K_n = 0$ for all sufficiently large n. Let k be a homogeneous element of K of degree d. Then k vanishes in $K(d)_{(x_i)}$ for all i, so for each i there is an integer ℓ_i such that $x_i^\ell k = 0$ for all $\ell \ge \ell_i$. If $x_0^{e_0} \cdots x_r^{e_r}$ is a monomial of degree $e_0 + \dots + e_r \ge \sum \ell_i - r$, then some e_i will satisfy $e_i \ge \ell_i$, and therefore this monomial kills k. Let $\ell(k) = \sum \ell_i - r$. Then $S_n k = 0$ for all $n \ge \ell(k)$.

Now let $k_1, \dots, k_j$ be a generating set for K, with k_i homogeneous of degree q_i for all i. Then $K_n = 0$ for all $n \ge \max\{q_i + \ell(k_i)\}$, so α_n is injective for these n, as was to be shown.

Alternate proof: As in the proof of (III, 5.2), we immediately reduce to the case $X = \mathbb{P}^r_A$. (This also uses the facts that $i_*\widetilde{M} \cong ({}_{A[x_0,\dots,x_r]}M)^\sim$, and that the maps α are compatible.)

Let $m_1, \dots, m_r$ be a finite homogeneous generating set for M, and for all i let q_i be minus the degree of m_i. Then we have a surjection $\bigoplus S(q_i) \to M$ of graded

S-modules (preserving degrees). Let K be the kernel of this map, and again construct a degree-preserving surjection $\bigoplus S(r_j) \to K$ of graded S-modules. We then have an exact sequence

$$\bigoplus S(r_j) \longrightarrow \bigoplus S(q_i) \longrightarrow M \longrightarrow 0$$

of graded S-modules. The corresponding sequence of sheaves is also exact, by (II 5.11a) and exactness of localization.

Consider the following commutative diagram.

$$\begin{array}{ccccccc} \bigoplus S_{r_j+d} & \longrightarrow & \bigoplus S_{q_i+d} & \longrightarrow & M_d & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \alpha_d} & & \\ \Gamma(X, \bigoplus \mathscr{O}(r_j+d)) & \longrightarrow & \Gamma(X, \bigoplus \mathscr{O}(q_i+d)) & \longrightarrow & \Gamma(X, \widetilde{M}(d)) & \longrightarrow & 0 \end{array}$$

As noted above, the top row is exact. We claim that the bottom row is also exact for all sufficiently large d. Indeed, let K be as above, and let $K' = \ker(\bigoplus S(r_j) \to K)$. Then the second map in the bottom row is surjective for all d such that $H^1(X, \widetilde{K}(d)) = 0$, and the first map in the bottom row maps onto the kernel $\widetilde{K}(d)$ of the second map for all d such that $H^1(X, \widetilde{K}'(d)) = 0$. These conditions are true for all $d \gg 0$, so the bottom row is exact for those values of d. (See also Ex. 5.10.)

The first two vertical arrows in the diagram are isomorphisms by (II, 5.13) or (III, 5.1a), and therefore so is the third vertical arrow, by the five-lemma.

2. (10 points) Hartshorne III Ex. 5.3: *Arithmetic genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathscr{O}_X) - 1)\ .$$

Note that it depends only on X, not on any projective embedding.

(a). If X is integral, and k algebraically closed, show that $H^0(X, \mathscr{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathscr{O}_X)\ .$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X, \mathscr{O}_X)\ .$$

[*Hint*: Use (I, 3.4).]

(b). If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c). If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

In this solution, we use the common notation $h^i(X,\mathscr{F}) = \dim_k H^i(X,\mathscr{F})$.

(a). If X is integral and k is algebraically closed, then X is a variety, and by (I Thm. 3.4a) $H^0(X,\mathscr{O}_X) = \mathscr{O}_X(X) = k$. Thus $h^0(X,\mathscr{O}_X) = 1$,

$$\chi(X) - 1 = \sum_{i=1}^{r}(-1)^i h^i(X,\mathscr{O}_X) ,$$

and

$$p_a(X) = \sum_{i=1}^{r}(-1)^{r-i} h^i(X,\mathscr{O}_X) = \sum_{i=0}^{r-1}(-1)^i h^{r-i}(X,\mathscr{O}_X) .$$

If X is a curve, then $r = 1$ and we just get $p_a(X) = h^1(X,\mathscr{O}_X)$.

(b). Let S_X be the homogeneous coordinate ring of X associated to its embedding in $\mathbb{P}^r_k$. Then $\mathscr{O}_X = \widetilde{S_X}$ and therefore $S_X \cong \Gamma_*(\mathscr{O}_X)$ by (II, Ex. 5.9b) (the proof from class works without the assumption that $X = \mathbb{P}^r_k$), so the Hilbert polynomial P_X of X equals the Hilbert polynomial of $\mathscr{O}_X$ defined in Ex. 5.2. Thus $P_X(0) = \chi(\mathscr{O}_X(0)) = \chi(\mathscr{O}_X)$. This suffices to show that the two definitions of arithmetic genus are the same in this case.

(c). For nonsingular projective curves, birational means isomorphic (I Cor. 6.12), so the first sentence is obvious. If X is a nonsingular plane curve of degree $d \ge 3$, then by (I Ex. 7.2), $p_a(X) \ne 0$. Since $p_a(\mathbb{P}^1) = h^1(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}) = h^0(\mathbb{P}^1,\mathscr{O}(-2)) = 0$, this leads to a contradiction if X is rational. Therefore X is not rational.

3. **[WITHDRAWN]**

4(NC). (10 points) Hartshorne III Ex. 5.10: Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \dots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \ge n_0$, the sequence of global sections

$$\Gamma(X,\mathscr{F}^1(n)) \to \Gamma(X,\mathscr{F}^2(n)) \to \dots \to \Gamma(X,\mathscr{F}^r(n))$$

is exact.

It will suffice to prove this when $r = 3$. Pick a very ample (or just ample) line sheaf $\mathscr{O}(1)$ on X, and let

$$\mathscr{F}^1 \xrightarrow{\phi} \mathscr{F}^2 \xrightarrow{\psi} \mathscr{F}^3$$

be an exact sequence of coherent sheaves on X. Let $\mathscr{G} = \operatorname{im}\phi = \ker\psi$; it is a subsheaf of $\mathscr{F}^2$. Pick n_0 such that $H^1(X,(\ker\phi)(n)) = 0$ for all $n \ge n_0$. For such n the long exact sequence associated to the short exact sequence

$$0 \to (\ker\phi)(n) \to \mathscr{F}^1(n) \to \mathscr{G}(n) \to 0$$

gives

$$H^0(X,\mathscr{F}^1(n)) \to H^0(X,\mathscr{G}(n)) \to 0 ; \tag{1}$$

in addition, the exact sequence

$$0 \longrightarrow \mathscr{G}(n) \longrightarrow \mathscr{F}^2(n) \longrightarrow \mathscr{F}^3(n)$$

gives an exact sequence

$$0 \longrightarrow H^0(X, \mathscr{G}(n)) \longrightarrow H^0(X, \mathscr{F}^2(n)) \longrightarrow H^0(X, \mathscr{F}^3(n)) \tag{2}$$

since the global section functor is left exact. Combining (1) and (2) then gives the exact sequence

$$H^0(X, \mathscr{F}^1(n)) \longrightarrow H^0(X, \mathscr{F}^2(n)) \longrightarrow H^0(X, \mathscr{F}^3(n)) \ ,$$

as was to be shown.