Math 256B. Solutions to Homework 7

1. (10 points) Show directly from the definition (without using cohomology) that if $i\colon X \to Y$ is a closed embedding of noetherian schemes and if $\mathscr{L}$ is an ample line sheaf on Y, then the line sheaf $i^*\mathscr{L}$ is ample on X.

   Let $\mathscr{F}$ be a coherent sheaf on X. Then, by II, Ex. 5.5, $i_*\mathscr{F}$ is coherent, so $i_*\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for sufficiently large n. But, by II, Ex. 5.1d, $i_*\mathscr{F} \otimes \mathscr{L}^n \cong i_*(\mathscr{F} \otimes i^*\mathscr{L}^n)$. Since i_* induces an isomorphism

$$\Gamma(X, \mathscr{F} \otimes i^*\mathscr{L}^n) \xrightarrow{\sim} \Gamma(Y, i_*(\mathscr{F} \otimes i^*\mathscr{L}^n)) ,$$

   and since i_* induces isomorphisms on stalks (II, Ex. 1.19a), it follows that $\mathscr{F} \otimes i^*\mathscr{L}^n$ is generated by global sections for all sufficiently large n. Thus $i^*\mathscr{L}$ is ample.

2. (15 points) Hartshorne II Ex. 7.5: Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}$ and $\mathscr{M}$ will denote invertible sheaves, and for (d) and (e) we assume furthermore that X is of finite type over a noetherian ring A.

   (a). If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is ample.
   (b). If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.
   (c). If $\mathscr{L}$, $\mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.
   (d). If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.
   (e). If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \ge n_0$.

   Do not use cohomology for this exercise.

   In part (d), for partial credit you may assume that X is proper over $\operatorname{Spec} A$, or for slightly more partial credit you may instead assume that X is reduced. For parts (d) and (e), "very ample" means very ample over $\operatorname{Spec} A$.

   (a). Let $\mathscr{F}$ be a coherent sheaf on X. Since $\mathscr{L}$ is ample, there is an integer n_0 such that $\mathscr{F} \otimes \mathscr{L}^n$ is gbgs for all $n \ge n_0$. We may assume that $n_0 \ge 0$. Then $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n \cong (\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^n$ is gbgs, because it is a tensor power of the gbgs sheaves $\mathscr{F} \otimes \mathscr{L}^n$ and $\mathscr{M}$ (with $\mathscr{M}$ being repeated $n \ge 0$ times).

   (b). Pick n_0 such that $\mathscr{M} \otimes \mathscr{L}^{n_0}$ is generated by global sections. For all $r \in \mathbb{Z}_{>0}$, $\mathscr{L}^r$ is ample, so $(\mathscr{M} \otimes \mathscr{L}^{n_0}) \otimes \mathscr{L}^r \cong \mathscr{M} \otimes \mathscr{L}^{n_0+r}$ is ample, by (a). Thus $\mathscr{M} \otimes \mathscr{L}^n$ is ample for all $n > n_0$.

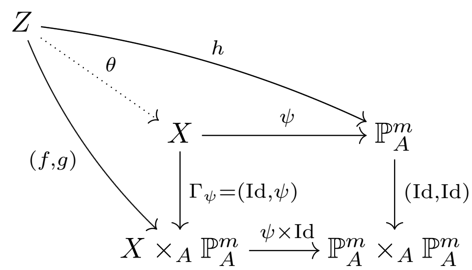
   (c). Since $\mathscr{M}$ is ample, there is an integer $n > 0$ such that $\mathscr{O}_X \otimes \mathscr{M}^n \cong \mathscr{M}^n$ is generated by global sections. By II, Prop. 7.5, $\mathscr{L}^n$ is ample; by part (a)

$$\mathscr{L}^n \otimes \mathscr{M}^n \cong (\mathscr{L} \otimes \mathscr{M})^n$$

   is ample; and then by Prop. 7.5 again, $\mathscr{L} \otimes \mathscr{M}$ is ample.

(d). Recall that "$\mathscr{L}$ is very ample" means that $\mathscr{L}$ is very ample over A. Let $s_0, \dots, s_n$ be global sections of $\mathscr{L}$ corresponding to an embedding $\phi\colon X \to \mathbb{P}^n_A$. Let U be an open subset of $\mathbb{P}^n_A$ such that $\phi\colon X \to U$ is a closed embedding (this exists by the definitions of embedding, closed embedding, and relative topology). Let $t_0, \dots, t_m$ be global sections of $\mathscr{M}$ generating it, and let $\psi\colon X \to \mathbb{P}^m_A$ be the corresponding morphism. We claim that $(\phi, \psi)\colon X \to U \times_A \mathbb{P}^m_A$ is a closed embedding.

We first show that the graph $\Gamma_\psi\colon X \to X \times_A \mathbb{P}^m_A$ is a closed embedding. (This was noted in class on Friday, 15 March, but that was after this assignment was due.) Since $\mathbb{P}^m_A$ is separated over $\operatorname{Spec} A$, the diagonal map $(\mathrm{Id}, \mathrm{Id})$ is a closed embedding, and it will suffice to show that the square in the diagram



is cartesian (i.e., that $X = (X \times_A \mathbb{P}^m_A) \times_{\mathbb{P}^m_A \times_A \mathbb{P}^m_A} \mathbb{P}^m_A$). To see this, it is first of all easy to check that the square commutes. Also, commutativity of the perimeter implies that $(\psi \circ f, g) = (h, h)$, and therefore $\psi \circ f = g = h$. The commutativity conditions on θ are $(\theta, \psi \circ \theta) = (f, g)$ and $\psi \circ \theta = h$. The first condition implies $\theta = f$, so any θ is unique. This value of θ satisfies the commutativity conditions since $\psi \circ f = g = h$, so the square is indeed cartesion, as was to be shown.

By base change, the morphism $\phi \times \mathrm{Id}_{\mathbb{P}^m_A}\colon X \times_A \mathbb{P}^m_A \to U \times_A \mathbb{P}^m_A$ is a closed embedding. Composing this with the closed embedding Γ_ψ, we find that $(\phi, \psi)\colon X \to U \times_A \mathbb{P}^m_A$ is a closed embedding.

We now claim that $(\phi, \psi)\colon X \to \mathbb{P}^n_A \times_A \mathbb{P}^m_A$ is an embedding. Indeed, if X is proper over $\operatorname{Spec} A$, then we can take $U = \mathbb{P}^n_A$ above, and this is immediate. If X is reduced, then let $\overline{X}$ be the closure of $\operatorname{im}(\phi, \psi)$ with reduced induced subscheme structure, and then X is isomorphic to an open subscheme of the closed subscheme $\overline{X}$ of $\mathbb{P}^n_A \times_A \mathbb{P}^m_A$. In general, one uses II, Ex. 3.11d (see Görtz–Wedhorn (10.8)).

Composing this embedding with the Segre embedding $\sigma\colon \mathbb{P}^n_A \times_A \mathbb{P}^m_A \to \mathbb{P}^{nm+n+m}_A$ (which is a closed embedding), we find that

$$\sigma \circ (\phi, \psi)\colon X \to \mathbb{P}^{nm+n+m}_A$$

is an embedding. Since this map corresponds to the global sections $s_i t_j$, $0 \le i \le n$, $0 \le j \le m$ of $\mathscr{L} \otimes \mathscr{M}$, it follows that $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(e). Pick n' such that $\mathscr{L}^{n'}$ is very ample, and pick n'' such that $\mathscr{L}^n$ is generated by global sections for all $n \ge n''$. Then $\mathscr{L}^n \cong \mathscr{L}^{n'} \otimes \mathscr{L}^{n-n'}$ is very ample for all $n \ge n' + n''$, by part (d).

3. (15 points) Hartshorne III Ex. 5.7(a)–(c): Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.
   (a). If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i\colon Y \to X$ is the inclusion. **Added condition:** Use cohomology for this part.
   (b). $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes \mathscr{O}_{X_{\text{red}}}$ is ample on X_{red}.
   (c). Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathscr{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(a). Let $\mathscr{L}$ be ample on X, and let $\mathscr{F}$ be any coherent sheaf on Y. Then $i_*\mathscr{F}$ is coherent (II Ex. 5.5), and by Lemma 2.10, (II Ex. 6.8a), (II Ex. 5.1d), and III, Prop. 5.3, we have

$$H^i(Y, \mathscr{F} \otimes (i^*\mathscr{L})^n) = H^i(X, i_*(\mathscr{F} \otimes (i^*\mathscr{L})^n)) = H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n) = 0$$

for all $n \gg 0$. Thus $i^*\mathscr{L}$ is ample on Y, by Prop. 5.3 again.

(b). Since $i\colon X_{\text{red}} \to X$ is a closed embedding and $\mathscr{L}_{\text{red}} = i^*\mathscr{L}$ (II Prop. 5.2), one implication is immediate from part (a).

Conversely, suppose that $\mathscr{L}_{\text{red}}$ is ample, and let $\mathscr{F}$ be a coherent sheaf on X. Let $\mathscr{N}$ be the sheaf of nilpotent elements on X, so that $\mathscr{O}_{X_{\text{red}}} \cong \mathscr{O}_X/\mathscr{N}$, and recall from the proof of III, Ex. 3.1 (on Homework 3) that $\mathscr{N}^r = 0$ for some $r \in \mathbb{N}$. Then, for $0 \le j < r$, the sheaf $\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}$ is a coherent sheaf on X_{red}, so $H^i(X_{\text{red}}, (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes_{\mathscr{O}_{X_{\text{red}}}} \mathscr{L}_{\text{red}}^n) = 0$ for all $i > 0$ and all $n \ge n_0(j)$ for some $n_0(j) \in \mathbb{Z}$. By elementary properties of tensor, we have that

$$(\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes_{\mathscr{O}_{X_{\text{red}}}} \mathscr{L}_{\text{red}}^n \cong (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{L}^n$$

for all n, and the latter can be regarded as a sheaf on X. Thus, by III, Lemma 2.10,

$$H^i(X, (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes \mathscr{L}^n) = 0$$

for all $i > 0$, all $n \ge n_0(j)$, and all $0 \le j < r$. Let $n_0 = \max\{n_0(0), \dots, n_0(r-1)\}$.

From the short exact sequence

$$0 \longrightarrow \mathscr{N}^{j+1}\mathscr{F} \otimes \mathscr{L}^n \longrightarrow \mathscr{N}^j\mathscr{F} \otimes \mathscr{L}^n \longrightarrow (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes \mathscr{L}^n \longrightarrow 0$$

we get an exact sequence

$$H^i(X, \mathscr{N}^{j+1}\mathscr{F} \otimes \mathscr{L}^n) \longrightarrow H^i(X, \mathscr{N}^j\mathscr{F} \otimes \mathscr{L}^n) \longrightarrow H^i(X, (\mathscr{N}^j\mathscr{F}/\mathscr{N}^{j+1}\mathscr{F}) \otimes \mathscr{L}^n)\,.$$

By descending induction on j, and noting that $\mathscr{N}^r\mathscr{F} = 0$, so $H^i(X, \mathscr{N}^r\mathscr{F} \otimes \mathscr{L}^n) = 0$ for all i and all n, it follows from the above sequence that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $i > 0$ and all $n \ge n_0$, so $\mathscr{L}$ is ample on X by III, Prop. 5.3.

(c). Again, since $j_i\colon X_i \to X$ is a closed embedding and $\mathscr{L} \otimes \mathscr{O}_{X_i} = j_i^*\mathscr{L}$ for all i, one implication is immediate from part (a) (and III, Lemma 2.10).

Conversely, suppose that $\mathscr{L} \otimes \mathscr{O}_{X_i}$ is ample on X_i for all i. It will suffice to prove the following lemma.

Lemma. *Let X be a reduced scheme, proper over A, let Y and Z be reduced closed subschemes of X such that $X = Y \cup Z$, and let $\mathscr{L}$ be a line sheaf on X such that $i^*\mathscr{L}$ and $j^*\mathscr{L}$ are ample on Y and Z, respectively, where $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$ are the inclusion maps. Then $\mathscr{L}$ is ample (on X).*

Proof. Let $\mathscr{I}$ and $\mathscr{J}$ be the sheaves of ideals corresponding to Y and Z, respectively. Then $\mathscr{I}\mathscr{J} \subseteq \mathscr{I} \cap \mathscr{J} = 0$, since $\mathscr{I} \cap \mathscr{J}$ is associated to a closed subscheme of X whose underlying topological space is all of X, and since X is reduced.

Let $\mathscr{F}$ be a coherent sheaf on X. Consider the filtration

$$\mathscr{F} \supseteq \mathscr{I}\mathscr{F} \supseteq \mathscr{I}\mathscr{J}\mathscr{F} = 0 .$$

The sheaf $\mathscr{I}\mathscr{F}$ is killed by $\mathscr{J}$, as is $\mathscr{I}\mathscr{F} \otimes \mathscr{L}^n$ for all n. Therefore, by (II, 5.2), $\mathscr{I}\mathscr{F} \otimes \mathscr{L}^n \cong j_*j^*(\mathscr{I}\mathscr{F} \otimes \mathscr{L}^n)$ for all n; note that $j^*(\mathscr{I}\mathscr{F})$ is a coherent sheaf on Z. Therefore, by III, Lemma. 2.10 and III, Prop. 5.3,

$$\begin{aligned} H^p(X, \mathscr{I}\mathscr{F} \otimes \mathscr{L}^n) &\cong H^p(X, j_*j^*(\mathscr{I}\mathscr{F} \otimes \mathscr{L}^n)) \\ &\cong H^p(Z, j^*(\mathscr{I}\mathscr{F} \otimes \mathscr{L}^n)) \cong H^p(Z, j^*(\mathscr{I}\mathscr{F}) \otimes (j^*\mathscr{L})^n) = 0 \end{aligned}$$

for all $p > 0$ and all $n \gg 0$ (depending only on $\mathscr{F}$ and $\mathscr{I}$). Next consider the quotient sheaf $\mathscr{G} := \mathscr{F}/\mathscr{I}\mathscr{F}$. It is killed by $\mathscr{I}$, so by a similar argument $H^p(X, \mathscr{G} \otimes \mathscr{L}^n) = 0$ for all $p > 0$ and all $n \gg 0$ depending only on $\mathscr{G}$ (and Y).

Now twist the short exact sequence

$$0 \longrightarrow \mathscr{I}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

by $\mathscr{L}^n$ and take the long exact sequence in cohomology. This gives (in part)

$$H^p(X, \mathscr{I}\mathscr{F} \otimes \mathscr{L}^n) \longrightarrow H^p(X, \mathscr{F} \otimes \mathscr{L}^n) \longrightarrow H^p(X, \mathscr{G} \otimes \mathscr{L}^n)$$

for all $p > 0$. For all $n \gg 0$ the two end terms are zero, so the middle term must also be zero. By III, Prop. 5.3 it then follows that $\mathscr{L}$ is ample. $\square$