Math 256B. Solutions to Homework 8

1(NC). (10 points) Let $\mathscr{L}$ be a line sheaf on a scheme X. Suppose that there exist global sections $s \in \Gamma(X, \mathscr{L})$ and $t \in \Gamma(X, \mathscr{L}^\vee)$ such that $s \otimes t$ maps to 1 under the canonical isomorphism $\mathscr{L} \otimes \mathscr{L}^\vee \xrightarrow{\sim} \mathscr{O}_X$. Show that there exist isomorphisms $\phi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{O}_X$ and $\psi\colon \mathscr{L}^\vee \xrightarrow{\sim} \mathscr{O}_X$ such that $\phi(s) = \psi(t) = 1$.

We first claim that s generates $\mathscr{L}$. Suppose not. Then $s_x \in \mathfrak{m}_x \mathscr{L}_x$ for some $x \in X$. Let σ be a generator for $\mathscr{L}_x$. Then $s = f\sigma$ for some $f \in \mathfrak{m}_x$. Letting $\alpha\colon \mathscr{L} \otimes \mathscr{L}^\vee \xrightarrow{\sim} \mathscr{O}_X$ denote the canonical isomorphism, we have that

$$1 = \alpha_x(s_x \otimes t_x) = f\alpha_x(\sigma \otimes t_x)$$

with $f \in \mathfrak{m}_x$, contradicting the fact that 1 generates $\mathscr{O}_X$ at x. Thus s generates $\mathscr{L}$, so there is an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{L}$ defined by $1 \mapsto s$. The inverse of this isomorphism is the desired isomorphism ϕ.

Similarly, it will suffice to show that t generates $\mathscr{L}^\vee$. If it does not, then $t_x = f\tau$ for some $f \in \mathfrak{m}_x$, $\tau \in \mathscr{L}_x^\vee$, and $x \in X$. Again,

$$1 = \alpha_x(s_x \otimes t_x) = f\alpha_x(s_x \otimes \tau) ,$$

contradicting the fact that 1 generates $\mathscr{O}_X$ at x. Thus t generates $\mathscr{L}^\vee$, and we conclude as before.

2. (10 points) Let $\mathscr{L}$ be an ample line sheaf on a projective variety X. Let $V_1, \dots, V_n$ be closed subvarieties, with $V_i \nsubseteq V_j$ for all $i \ne j$. Then there exists a positive integer m and $t_1, \dots, t_n \in \Gamma(X, \mathscr{L}^m)$ such that $t_i\big|_{V_j} \ne 0$ for all $i \ne j$, and $t_i\big|_{V_i} = 0$ for all i.

Here if X is a scheme, if $\mathscr{M}$ is a quasi-coherent sheaf on X, if Y is a closed subscheme of X with corresponding closed immersion $i\colon Y \to X$, and if s is a global section of $\mathscr{M}$, then $s\big|_Y$ is the global section of $i^*\mathscr{M}$ defined locally by $s \otimes 1$ (via the isomorphism of (II, Prop. 5.2e)). Or, tensoring the natural map $\mathscr{O}_X \to \mathscr{O}_Y$ with $\mathscr{M}$ gives a map $\mathscr{M} \to \mathscr{M} \otimes \mathscr{O}_Y$, and $s\big|_Y$ is the image of s under this map (note that $\mathscr{M} \otimes \mathscr{O}_Y \cong i^*\mathscr{M}$). It is also true (and you may use without proof) that if Y is integral, and if η is its generic point, then $s\big|_Y = 0$ if and only if $s_\eta \in \mathfrak{m}_\eta \mathscr{M}_\eta$, where $\mathfrak{m}$ is the maximal ideal in the local ring $\mathscr{O}_{X,\eta}$.

[**Hint:** Think of the prime avoidance lemma in commutative algebra.]

The $n = 1$ case of this problem is trivial (take $t_1 = 0$ and any $m > 0$).

Next, we consider the case $n = 2$. Let $\mathscr{I}_2$ be the ideal sheaf of V_2, and let $i\colon V_2 \to X$ be the closed embedding corresponding to V_2. We have a short exact sequence

$$0 \longrightarrow \mathscr{I}_2 \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_{V_2} \longrightarrow 0 .$$

This sequence remains exact after tensoring with $\mathscr{L}^m$ for any m:

$$0 \longrightarrow \mathscr{I}_2 \otimes_{\mathscr{O}_X} \mathscr{L}^m \xrightarrow{\alpha} \mathscr{L}^m \longrightarrow i_*\mathscr{O}_{V_2} \otimes_{\mathscr{O}_X} \mathscr{L}^m \longrightarrow 0 .$$

Therefore, for all $x \in X$, taking stalks at x also gives an exact sequence:

$$0 \longrightarrow (\mathscr{I}_2 \otimes_{\mathscr{O}_X} \mathscr{L}^m)_x \xrightarrow{\alpha_x} \mathscr{L}^m_x \longrightarrow (i_*\mathscr{O}_{V_2} \otimes_{\mathscr{O}_X} \mathscr{L}^m)_x \longrightarrow 0 .$$

If $x \notin V_2$, then $(i_*\mathscr{O}_{V_2})_x = 0$ because $x \notin V_2$, so $i_*\mathscr{O}_{V_2}(U) = 0$ for all open U with $x \in U \subseteq X \setminus V_2$. Therefore α_x is an isomorphism for such x. Moreover, since $\mathscr{L}^m$ is a line sheaf, $(\mathscr{I}_2 \otimes \mathscr{L}^m)_x \cong \mathscr{O}_{X,x}$ (non-canonically), and any set of sections of $\mathscr{I}_2 \otimes \mathscr{L}^m$ which generates the sheaf in a neighborhood of x must contain at least one element not in $\mathfrak{m}_x(\mathscr{I}_2 \otimes \mathscr{L}^m)_x$, and conversely any such element generates the sheaf in some (possibly smaller) open neighborhood of the point.

In particular, this is true at the generic point η_1 of V_1, since $V_1 \nsubseteq V_2$.

Since $\mathscr{I}_2$ is coherent and $\mathscr{L}$ is ample, there is an integer m_0 such that $\mathscr{I}_2 \otimes \mathscr{L}^m$ is generated by global sections for all $m \geq m_0$. Fix such an m. Then there is a global section $t_0 \in \Gamma(X, \mathscr{I}_2 \otimes \mathscr{L}^m)$ which generates the sheaf at η_1. Let $t = \alpha(t_0)$. Then t is a global section of $\mathscr{L}^m$, $t \notin \mathfrak{m}_{\eta_1}\mathscr{L}^m_{\eta_1}$ (because α_{η_1} is an isomorphism), and therefore $t\big|_{V_1} \neq 0$. But (since t comes from t_0), it lies in the kernel of the map to $i_*\mathscr{O}_{V_2}$, so $t\big|_{V_2} = 0$. This is the desired section, and the case $n = 2$ follows. (Actually, only half of it is proved, but the other half is true by symmetry.) Note that the result here is actually proved for all m such that $\mathscr{I}_1 \otimes \mathscr{L}^m$ and $\mathscr{I}_2 \otimes \mathscr{L}^m$ are generated by global sections.

The proof of the general case then follows by an argument similar to that used for (the easy version of) the prime avoidance lemma. Indeed, pick ℓ such that $\mathscr{I}_i \otimes \mathscr{L}^\ell$ is generated by global sections for all i. By the previous case, for all $i \neq j$ we may choose $t_{ij} \in \mathscr{L}^\ell$ such that $t_{ij}\big|_{V_i} \neq 0$ but $t_{ij}\big|_{V_j} = 0$. Then, for all i,

$$s_i := \bigotimes_{j \neq i} t_{ij} \in \Gamma(X, \mathscr{L}^{\ell(n-1)})$$

satisfies $s_i\big|_{V_i} \neq 0$ but $s_i\big|_{V_j} = 0$ for all $j \neq i$. Thus $t_i := \sum_{j \neq i} s_j$ satisfies the condition of the problem.

Alternate, shorter proof: Taking a multiple of $\mathscr{L}$, we may assume that $\mathscr{L}$ is very ample, corresponding to an embedding of X into $\mathbb{P}^N$. Let H be a hyperplane in $\mathbb{P}^N$ which does not contain any of the X_i (a suitably generic choice suffices). Then $X' := X \setminus (X \cap H)$ is an open affine subset of X which meets all of the X_i. Let $A = \mathscr{O}_X(X')$ and for all i let $\mathfrak{a}_i$ be the ideal of X_i. Then for each $i \neq j$ there exists $a_{ij} \in \mathfrak{a}_i$ such that $a_{ij} \notin \mathfrak{a}_j$. Letting $b_j = \prod_{i \neq j} a_{ij}$, we have $b_j \in \mathfrak{a}_i$ if and only if $i \neq j$. (Here we use the fact that the X_i are integral.) Then $c_i = \sum_{j \neq i} b_j$ has the property that $c_i \in \mathfrak{a}_j$ if and only if $i = j$. By (II, 5.14b), these c_i extend to sections $t_i \in \Gamma(X, \mathscr{L}^m)$ for some $m > 0$.

3. (15 points) Let A_0 be the subring of $k[x,y]$ generated by the set of all homogeneous polynomials of degree $\ne 1$, let $A = (A_0)_{x^2-1}$, and let $X = \operatorname{Spec} A$.
   (a). Show that X is separated, noetherian, integral, and regular in codimension one.
   (b). Show that X is not normal.
   (c). Show that the divisor $(x-1)$ equals zero as a Weil divisor, but that $x-1$ is not a regular function on X. (Note that $x-1$ is an element of $K(X) = k(x,y)$.)

(a). The scheme X is separated because it is affine. The ring A_0 is entire because it is a subring of the field $k(x,y)$, so X is integral. Also, A_0 is generated as a k-algebra by $x^2, xy, y^2, x^3, x^2y, xy^2, y^3$, so it is noetherian; hence A and X are also noetherian.

Now let $\mathfrak{m}$ be the ideal $(x^2, xy, y^2, x^3, x^2y, xy^2, y^3)$ in A_0. Then $A_0/\mathfrak{m} \cong k$, so $\mathfrak{m}$ is a maximal ideal. By (I Thm. 1.8A), $\dim A_0 = 2$ and $\mathfrak{m}$ has height 2, so removing the corresponding point from $\operatorname{Spec} A_0$ does not affect whether it is nonsingular in codimension one. But now $X \setminus \{\mathfrak{m}\}$ is covered by the sets $D(f)$, as f ranges over $\{x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$, and for each such f the localized ring $(A_0)_f$ equals one of $k[x,y]_x$, $k[x,y]_y$, or $k[x,y]_{xy}$ (in $k(x,y) = \operatorname{Frac} A_0$). The spectrum of each of these rings is regular in codimension one, so $\operatorname{Spec} A_0$ and hence X is regular in codimension one.

(b). The scheme X is not normal because A is not integrally closed. To see the latter, note that x is integral over A (since $x^2 \in A_0$). However, if x was in A, then $x(x^2-1)^n$ would have to lie in A_0 for some $n \in \mathbb{N}$, but this cannot happen because it has a nonzero linear term. Thus A_0 is not integrally closed, so X is not normal.

(c). To evaluate $(x-1)$ as a Weil divisor, it suffices to work on $(\operatorname{Spec} A_0) \setminus \{\mathfrak{m}\}$. For each of the localizations $D(f)$ mentioned in the proof of part (a), $(x-1)$ is the prime divisor $x=1$, but that line has been removed from $\operatorname{Spec} A$, so $(x-1)$ is zero as a divisor on X.

Showing that $x-1$ is not in $\Gamma(X, \mathscr{O}_X)$ is equivalent to showing that $x-1 \notin A$. But if $x-1 \in A$ then $x \in A$, which was shown not to hold in the proof of part (b).