Math 256B. Solutions to Homework 9

1. (15 points) Hartshorne III Ex. 5.8: Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.
   (a). If X is irreducible and nonsingular, then X is projective by (II, 6.7).
   (b). If X is integral, let $\widetilde{X}$ be its normalization (II, Ex. 3.8). Show that $\widetilde{X}$ is complete and nonsingular, hence projective by (a). Let $f\colon \widetilde{X} \to X$ be the projection. Let $\mathscr{L}$ be a very ample invertible sheaf on $\widetilde{X}$. Show that there is an effective divisor $D = \sum P_i$ on $\widetilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$. Then use (Ex. 5.7d), (II, 7.6), and (II, 5.16.1) to show that X is projective.
   (c). If X is reduced, but not necessarily irreducible, let $X_1, \dots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.
   (d). Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{\mathrm{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

(a). By (II, 6.11.1A), nonsingular implies reduced. Everything else in this part is trivial.

(b). $\widetilde{X}$ is nonsingular since it is normal and of dimension one (I, Thm. 6.2A). Also $\widetilde{X}$ is complete since $\widetilde{X} \to X$ is finite, hence proper (II, Ex. 4.1), so the composition $\widetilde{X} \to X \to k$ is proper. Thus, by (a), $\widetilde{X}$ is projective.

We have that f is an isomorphism outside of a finite set $S \subseteq \widetilde{X}$, so pick a section $\sigma \in \Gamma(\widetilde{X}, \mathscr{L})$ which is nonzero at all points of S (note that k is infinite). Let $D = (\sigma)$, and let $\mathscr{L}_0 = \mathscr{O}(f_*D)$, where f_*D is easy to define in this case. Then $f^*f_*D = D$, so $f^*\mathscr{L}_0 \cong \mathscr{L}$. By (III, Ex. 5.7d), $\mathscr{L}_0$ is ample on X, so X is projective over k by (II, 7.6) and (II, 5.16.1).

(c). Surjectivity of $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is equivalent by Ex. 4.5 to surjectivity of

$$H^1(X, \mathscr{O}_X^*) \to \bigoplus H^1(X_i, \mathscr{O}_{X_i}^*) = H^1\Big(X, \bigoplus j_{i*}\mathscr{O}_{X_i}^*\Big)\,,$$

where $j_i\colon X_i \to X$ is the inclusion map for all i. We have an exact sequence

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \bigoplus j_{i*}\mathscr{O}_{X_i}^* \longrightarrow \mathscr{R} \longrightarrow 0$$

where $\mathscr{R}$ is defined to be the cokernel. So, by the long exact sequence in cohomology, it will suffice to show that $H^1(X, \mathscr{R}) = 0$. But the support of $\mathscr{R}$ is a discrete set of points, so $H^1(X, \mathscr{R}) = 0$ by Lemma 2.10 and Theorem 2.7. By (b), each X_i has an ample sheaf $\mathscr{L}_i$, so the sheaf $\mathscr{L} \in \operatorname{Pic} X$ guaranteed by the above surjection is ample by Ex. 5.7c. Thus X is projective by (II, 7.6) and (II, 5.16.1).

(d). Let $\mathscr{N}$ be the sheaf of nilpotent elements on X. Pick n such that $\mathscr{N}^n = 0$; this exists since X is noetherian. Pick n' such that $n' < n \le 2n'$, let $\mathscr{I} = \mathscr{N}^{n'}$, and let X' be the corresponding closed subscheme of X. Since $\mathscr{I}^2 = 0$, Ex. 4.6 applies and gives an exact sequence

$$\operatorname{Pic} X \longrightarrow \operatorname{Pic} X' \longrightarrow H^2(X, \mathscr{I}) .$$

But $H^2(X, \mathscr{I}) = 0$ by Theorem 2.7, so $\operatorname{Pic} X$ maps onto $\operatorname{Pic} X'$. But now if $\mathscr{N}'$ is the sheaf of nilpotent elements of X', then $\mathscr{N}'^{n'} = 0$, so by induction on n (as above) we find that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective. Then, by Ex. 5.7b and part (c), there is an ample line sheaf $\mathscr{L}$ on X, hence X is projective by (II, 7.6) and (II, 5.16.1).

2. (10 points) Let X be a projective scheme over a field k, and let $\mathscr{L}$ be a very ample line sheaf on X (over k). Show that there exists an integer m_0 such that the map

$$H^0(X, \mathscr{L}^{\otimes m})^{\otimes n} \longrightarrow H^0(X, \mathscr{L}^{\otimes mn})$$

is surjective for all $m \ge m_0$ and all $n \in \mathbb{Z}_{>0}$.

[**Hint:** Look at the case $X = \mathbb{P}^r_k$ first.]

Let $i \colon X \hookrightarrow \mathbb{P}^r_k$ be an embedding for which $i^*\mathscr{O}(1) \cong \mathscr{L}$, and pick m_0 such that the map $H^0(\mathbb{P}^r_k, \mathscr{O}(m)) \to H^0(X, \mathscr{L}^{\otimes m})$ is surjective for all $m \ge m_0$ (by the "major corollary" given in class on 28 February). The result then follows from the fact that $S_m^{\otimes n} \to S_{mn}$ is surjective for all $m, n \in \mathbb{N}$ (where S_d is the degree-d part of the graded ring $S := k[x_0, \dots, x_r]$):

$$\begin{array}{ccc} S_m^{\otimes n} = H^0(\mathbb{P}^r_k, \mathscr{O}(m))^{\otimes n} & \twoheadrightarrow & S_{mn} = H^0(\mathbb{P}^r_k, \mathscr{O}(mn)) \\ \downarrow & & \downarrow \\ H^0(X, \mathscr{L}^{\otimes m})^{\otimes n} & \twoheadrightarrow & h^0(X, \mathscr{L}^{\otimes mn}) \end{array}$$

3. (10 points) Hartshorne II Ex. 6.1. Also say what the isomorphism is.

Applying (6.6) n times gives that $X \times \mathbb{A}^n$ satisfies (*) and $\operatorname{Cl}(X \times \mathbb{A}^n) \cong \operatorname{Cl} X$. Since the map $X \times \mathbb{P}^n \to \operatorname{Spec} \mathbb{Z}$ factors as $X \times \mathbb{P}^n \to X \to \operatorname{Spec} \mathbb{Z}$ and both factors are separated (the first map is projective, hence separated), $X \times \mathbb{P}^n$ is separated. The scheme $X \times \mathbb{P}^n$ is irreducible, because it is covered by the irreducible open subsets $X \times D_+(x_i) \cong X \times \mathbb{A}^n$, and the generic points of these subsets are all the same. All other parts of (*) are local properties, so $X \times \mathbb{P}^n$ satisfies (*) (showing that $X \times \mathbb{P}^n$ is noetherian uses the fact that the open cover is finite).

Now let H be the hyperplane $x_0 = 0$ in $\mathbb{P}^n$. Then, letting $U = (X \times \mathbb{P}^n) \setminus (X \times H)$, the exact sequence

$$\mathbb{Z} \to \operatorname{Cl}(X \times \mathbb{P}^n) \to \operatorname{Cl}(U) \to 0$$

of (6.5) becomes

$$\mathbb{Z} \to \operatorname{Cl}(X \times \mathbb{P}^n) \to \operatorname{Cl}(X) \to 0 \qquad (\dagger)$$

by the above comments, since $U \cong X \times \mathbb{A}^n$.

Now the map on the left is injective. Indeed, restricting $n(X \times H)$ to $x \times \mathbb{P}^n$ for some closed point $x \in X$ implies that if $n(X \times H) \sim 0$ then $n = 0$, by (6.4). Also, we know the structure of the isomorphism (6.6); therefore the map $D \mapsto p^*(D)$ is a splitting of the above surjection, where $p\colon X \times \mathbb{P}^n \to X$ is the projection and p^* is defined by $p^*Y = p^{-1}(Y)$ for all prime divisors Y on X. This gives the desired product:

$$\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z} .$$

The isomorphism can be described by the map $\mathbb{Z} \to \mathrm{Cl}(X \times \mathbb{P}^n)$ given by $n \mapsto n \cdot H$, and the map $\mathrm{Cl}\, X \to \mathrm{Cl}(X \times \mathbb{P}^n)$ given by $D \mapsto p^*D$. Indeed, $n \mapsto n \cdot H$ is the first map in $(\dagger)$, and p^* splits the exact sequence.