Math 256B. A Lemma for the Solution of Hartshorne (III, Ex. 4.4)

Let X be a topological space, and let

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{R} \longrightarrow 0$$

be a short exact sequence of sheaves of abelian groups on X. Let $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{A}^{\cdot}$, $0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{B}^{\cdot}$, and $0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{C}^{\cdot}$ be resolutions of $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{R}$, respectively, such that $0 \longrightarrow \mathscr{A}^{\cdot} \longrightarrow \mathscr{B}^{\cdot} \longrightarrow \mathscr{C}^{\cdot}$ is exact (i.e., $0 \longrightarrow \mathscr{A}^i \longrightarrow \mathscr{B}^i \longrightarrow \mathscr{C}^i$ is exact for all i).

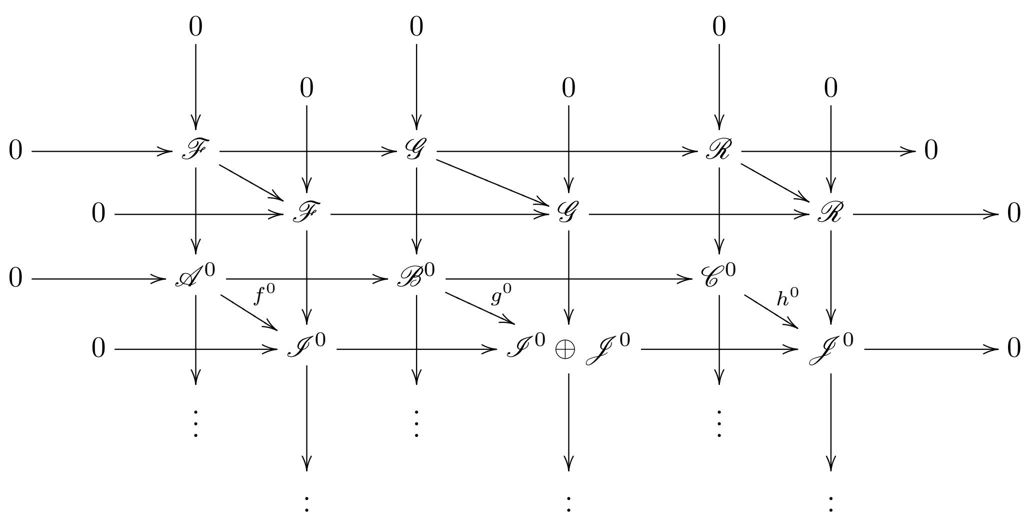
Also let $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^{\cdot}$ and $0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{J}^{\cdot}$ be injective resolutions of $\mathscr{F}$ and $\mathscr{R}$, respectively. Then, as was shown in class (on January 24, when proving Thm. 1.1A(c)) and on the first homework assignment, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{R} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I}^0 & \longrightarrow & \mathscr{I}^0 \oplus \mathscr{J}^0 & \longrightarrow & \mathscr{J}^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I}^1 & \longrightarrow & \mathscr{I}^1 \oplus \mathscr{J}^1 & \longrightarrow & \mathscr{J}^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \qquad (1)$$

with exact rows and columns.

We claim that there is a commutative diagram

(2)



with exact rows and columns given as above, and where the maps $\mathscr{F} \to \mathscr{F}$, $\mathscr{G} \to \mathscr{G}$, and $\mathscr{R} \to \mathscr{R}$ are the identity maps.

This is really just a matter of constructing the maps f^i, g^i, and h^i for all $i \geq 0$.

Lemma. *Assume that the diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{R} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{A}^0 & \longrightarrow & \mathscr{B}^0 & \longrightarrow & \mathscr{C}^0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{A}^1 & \longrightarrow & \mathscr{B}^1 & \longrightarrow & \mathscr{C}^1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3}$$

commutes and has exact rows and columns, and that we also have the diagram (1). Then there exist morphisms

$$f^{\cdot} : \mathscr{A}^{\cdot} \to \mathscr{I}^{\cdot}, \qquad g^{\cdot} : \mathscr{B}^{\cdot} \to (\mathscr{I} \oplus \mathscr{J})^{\cdot}, \qquad \text{and} \quad h^{\cdot} : \mathscr{C}^{\cdot} \to \mathscr{J}^{\cdot}$$

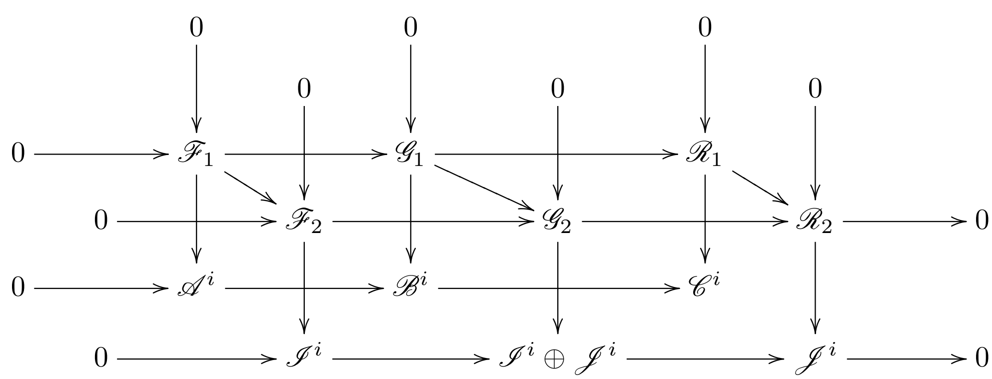
of complexes such that a diagram similar to (2) commutes.

Proof. In this proof we will first assume that $\mathscr{B}^{\cdot} \to \mathscr{C}^{\cdot}$ is surjective, and then later notice that this surjectivity assumption was never actually used.

Similar to the proof of Theorem 1.1A(c), we start by showing a "key step."

Claim. Let

(4)



be a commutative diagram with exact rows and columns. Then there exist morphisms $f^i\colon \mathscr{A}^i \to \mathscr{I}^i$, $g^i\colon \mathscr{B}^i \to \mathscr{I}^i \oplus \mathscr{J}^i$, and $h^i\colon \mathscr{C}^i \to \mathscr{J}^i$, such that the diagram still commutes.

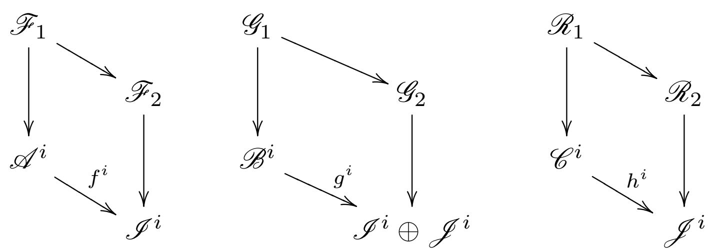
Proof. Fix a splitting of the sequence $0 \longrightarrow \mathscr{I}^i \longrightarrow \mathscr{I}^i \oplus \mathscr{J}^i \longrightarrow \mathscr{J}^i \longrightarrow 0$. In particular, this gives a projection $\mathscr{I}^i \oplus \mathscr{J}^i \to \mathscr{I}^i$. Composing this with the composed map $\mathscr{G}_1 \to \mathscr{G}_2 \to \mathscr{I}^i \oplus \mathscr{J}^i$ gives a map $\mathscr{G}_1 \to \mathscr{I}^i$. Since the map $\mathscr{G}_1 \to \mathscr{B}^i$ is an injection and $\mathscr{I}^i$ is an injective object, the map $\mathscr{G}^1 \to \mathscr{I}^i$ extends to give a map $\widetilde{f}^i\colon \mathscr{B}^i \to \mathscr{I}^i$ such that the composed map $\mathscr{G}^1 \to \mathscr{B}^i \xrightarrow{\widetilde{f}^i} \mathscr{I}^i$ equals the above map $\mathscr{G}^1 \to \mathscr{I}^i$.

Define $f^i\colon \mathscr{A}^i \to \mathscr{I}^i$ to be the composition $\mathscr{A}^i \to \mathscr{B}^i \xrightarrow{\widetilde{f}^i} \mathscr{I}^i$.

Similarly to the construction of f^i, we define $h^i\colon \mathscr{C}^i \to \mathscr{J}^i$ by extending the composition $\mathscr{R}_1 \to \mathscr{R}_2 \to \mathscr{J}^i$, using the facts that $\mathscr{J}^i$ is an injective object and $\mathscr{R}_1 \to \mathscr{C}^i$ is an injection. Composing h^i with the map $\mathscr{B}^i \to \mathscr{C}^i$ then gives a map $\widetilde{h}^i\colon \mathscr{B}^i \to \mathscr{J}^i$.

We then define $g^i\colon \mathscr{B}^i \to \mathscr{I}^i \oplus \mathscr{J}^i$ to be the sum $\widetilde{f}^i + \widetilde{h}^i$.

It remains to show that, after adding f^i, g^i, and h^i to the diagram (4), it still commutes. We first consider the parts



(5)

The rightmost piece commutes by construction of h^i.

We show that the middle piece commutes by arrow chasing, considering the components of $\mathscr{I}^i \oplus \mathscr{J}^i$ separately (using the chosen splitting).

Let $\gamma \in \mathscr{G}_1$, and let $\alpha^{\urcorner} + \beta^{\urcorner}$ and $\alpha_{\llcorner} + \beta_{\llcorner}$ denote its images in $\mathscr{I}^i \oplus \mathscr{J}^i$ via maps passing through $\mathscr{G}_2$ and $\mathscr{B}_i$, respectively, with $\alpha^{\urcorner}, \alpha_{\llcorner} \in \mathscr{I}^i$ and $\beta^{\urcorner}, \beta_{\llcorner} \in \mathscr{J}^i$.

We claim that $\beta^{\urcorner}$ is obtained from γ by each of the following compositions of maps:

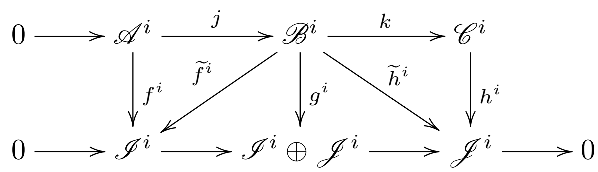
$$\mathscr{G}_1 \to \mathscr{G}_2 \to \mathscr{I}^i \oplus \mathscr{J}^i \to \mathscr{J}^i$$
$$\mathscr{G}_1 \to \mathscr{G}_2 \to \mathscr{R}_2 \to \mathscr{J}^i$$
$$\mathscr{G}_1 \to \mathscr{R}_1 \to \mathscr{R}_2 \to \mathscr{J}^i$$
$$\mathscr{G}_1 \to \mathscr{R}_1 \to \mathscr{C}^i \xrightarrow{h^i} \mathscr{J}^i$$
$$\mathscr{G}_1 \to \mathscr{B}^i \to \mathscr{C}^i \xrightarrow{h^i} \mathscr{J}^i .$$
$$\mathscr{G}_1 \to \mathscr{B}^i \xrightarrow{\widetilde{h}^i} \mathscr{J}^i .$$

Indeed, for the first composition, this is by definition of $\beta^{\urcorner}$. The second, third, and fifth follow from their predecessors by commutativity of (4), the fourth follows from its predecessor by commutativity of the rightmost of the three pieces of (5), and the last follows from its predecessor by definition of $\widetilde{h}^i$. But this latter element is actually $\beta_{\llcorner}$. Thus $\beta^{\urcorner} = \beta_{\llcorner}$.

We now consider $\alpha^{\urcorner}$ and $\alpha_{\llcorner}$. By definition, $\alpha_{\llcorner}$ is obtained by applying the composed map $\mathscr{G}_1 \to \mathscr{G}_2 \to \mathscr{I}^i \oplus \mathscr{J}^i \to \mathscr{I}^i$ to γ. By construction of $\widetilde{f}^i$, this map is equal to the composed map $\mathscr{G}_1 \to \mathscr{B}^i \xrightarrow{\widetilde{f}^i} \mathscr{I}^i$. Since applying this latter map to γ gives $\alpha_{\llcorner}$, we have $\alpha^{\urcorner} = \alpha_{\llcorner}$.

Therefore the center piece of (5) commutes.

Before showing that the leftmost piece of (5) commutes, we first show that the rectangles in the diagram



(6)

commute. (Note that the triangle with vertices $\mathscr{B}^i$, $\mathscr{I}^i$, and $\mathscr{I}^i \oplus \mathscr{J}^i$ does *not* commute, since that would require $\widetilde{h}^i$ to be the zero map.)

The triangles above and below the arrow $\widetilde{h}^i$ commute by the definitions of $\widetilde{h}^i$ and g^i, respectively. Therefore the rectangle on the right commutes.

Showing commutativity of the rectangle on the left is a matter of showing that $(\widetilde{f}^i + \widetilde{h}^i) \circ j = f^i$. Since $\widetilde{f}^i \circ j = f^i$, this is equivalent to $\widetilde{h}^i \circ j = 0$. This holds because $\widetilde{h}^i = h^i \circ k$ and $k \circ j = 0$.

Commutativity of the leftmost piece of (5) then follows immediately from commutativity of the center piece of (5), since the leftmost piece is obtained from the center piece by restricting the various maps to $\mathscr{F}_1$, $\mathscr{F}_2$, or $\mathscr{A}^i$, and noting that the relevant other parts of the diagram are now known to commute. □

One then concludes the lemma by induction on i. Indeed, for the base case we apply the claim, with $i = 0$, $\mathscr{F}_1 = \mathscr{F}_2 = \mathscr{F}$, $\mathscr{G}_1 = \mathscr{G}_2 = \mathscr{G}$, and $\mathscr{R}_1 = \mathscr{R}_2 = \mathscr{R}$, and with the maps $\mathscr{F}_1 \to \mathscr{F}_2$, $\mathscr{G}_1 \to \mathscr{G}_2$, and $\mathscr{R}_1 \to \mathscr{R}_2$ equal to the identity maps. For the inductive step, assume that $i > 0$, and assume that the desired diagram has been constructed up to (or, actually, down to) the $i - 1$ level. Then we may apply the claim with $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{R}_1$, and $\mathscr{R}_2$ equal to the cokernels of the maps in the corresponding column of the previous application of the claim. Maps between these sheaves exist by the universal property of cokernels. By exactness of the vertical sequences in the diagrams (1) and (3), these cokernels inject into the i level of the diagrams. This gives another diagram (4) at the i level, and the induction may proceed. □

This gives the lemma needed for Ex. 4.4.