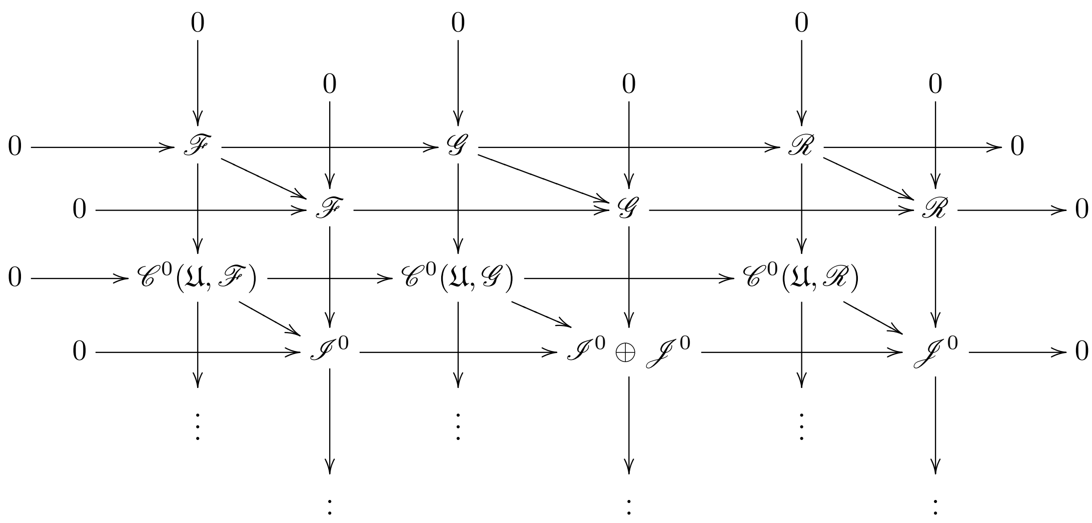
Math 256B. More Information on (III, Ex. 4.4) (revised)

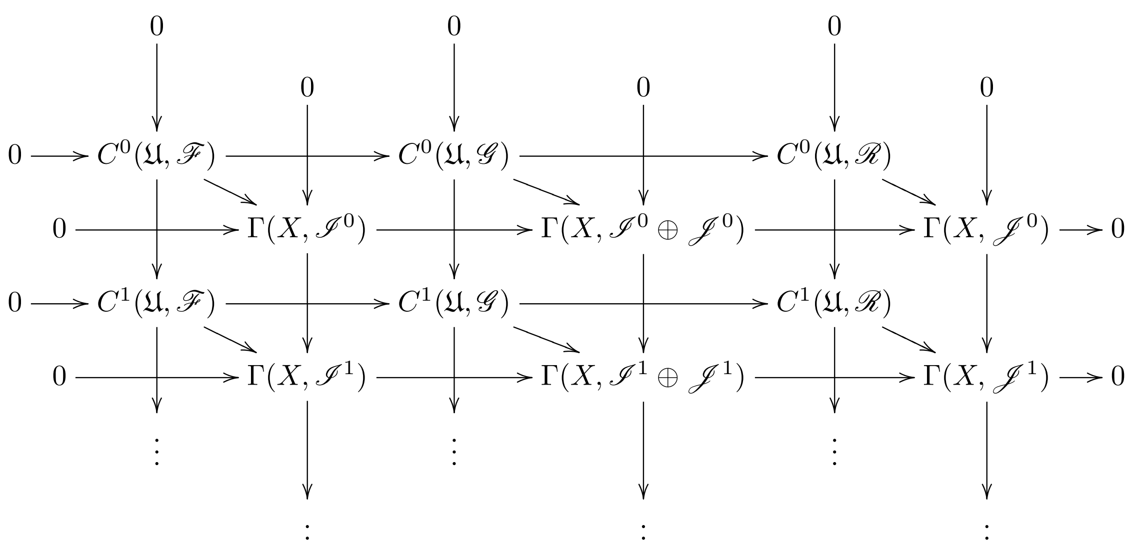
This handout provides more information on why the square (B) in the diagram (**) from class on February 21, commutes.

From the lemma in the previous handout and from class, there is a commutative diagram



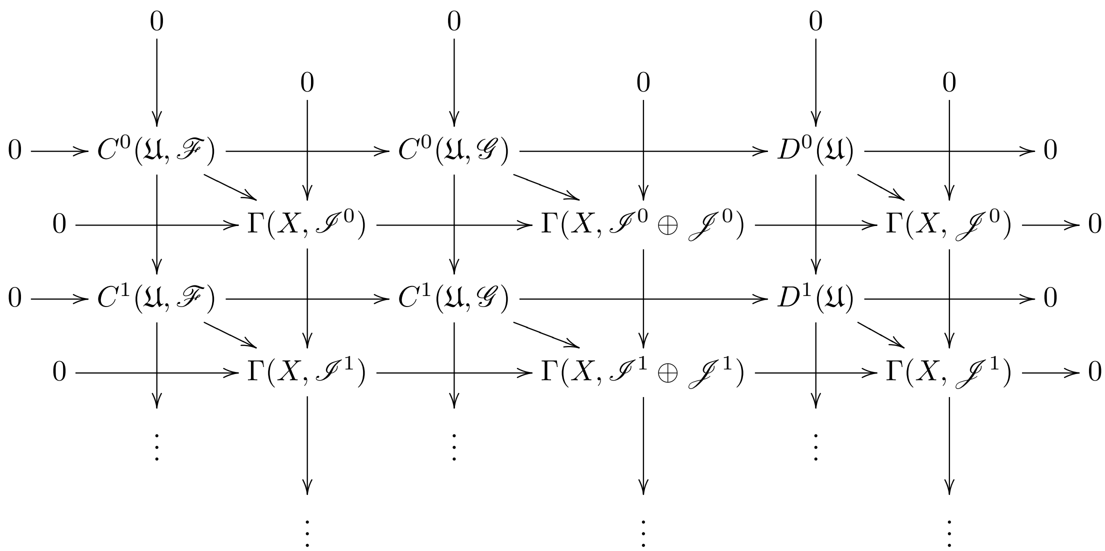
of sheaves on X, with exact rows and columns, where the maps $\mathscr{F} \to \mathscr{F}$, $\mathscr{G} \to \mathscr{G}$, and $\mathscr{R} \to \mathscr{R}$ are the identity maps.

Removing the plane containing the sheaves $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{R}$ and passing to global sections gives the commutative diagram



of abelian groups, with exact rows (because $\mathscr{I}^i$ is injective for all i), but whose columns are now complexes.

Finally, we replace $C^i(\mathfrak{U},\mathscr{R})$ with $D^i(\mathfrak{U})$ for all i, obtaining the following diagram, in which all rows are now short exact sequences and all columns are complexes:



This diagram consists of two sheets (joined by the diagonal arrows). Each sheet is a short exact sequence of complexes of abelian groups, and the Snake Lemma applied to each of these sheets gives long exact sequences of the top row of the diagram (*) and the bottom row of the diagram (**) from class, respectively:

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \Gamma(X,\mathscr{F}) & \longrightarrow & \Gamma(X,\mathscr{G}) & \longrightarrow & h^0(D^\cdot(\mathfrak{U})) & \xrightarrow{\delta} & \check{H}^1(\mathfrak{U},\mathscr{F}) & \longrightarrow & \check{H}^1(\mathfrak{U},\mathscr{G}) = 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle f_{\mathfrak{U}}} & & \downarrow{\scriptstyle g_{\mathfrak{U}}} & & \downarrow \\
0 \longrightarrow & \Gamma(X,\mathscr{F}) & \longrightarrow & \Gamma(X,\mathscr{G}) & \longrightarrow & \Gamma(X,\mathscr{R}) & \xrightarrow{\delta} & H^1(X,\mathscr{F}) & \longrightarrow & H^1(X,\mathscr{G}) = 0
\end{array}$$

The third square (between $f_{\mathfrak{U}}$ and $g_{\mathfrak{U}}$) commutes by naturality of the Snake Lemma, as sketched in class on February 21. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, then there is a commutative diagram (shaped like a Toblerone bar, with faces corresponding to the above diagram, the above diagram with $\mathfrak{U}$ replaced by $\mathfrak{V}$, and the diagram (*) from class), which allows us to take direct limits of the objects in the top row of the above diagram. This gives the following commutative diagram with exact rows, as was to be shown:

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \Gamma(X,\mathscr{F}) & \longrightarrow & \Gamma(X,\mathscr{G}) & \longrightarrow & \varinjlim h^0(D^\cdot(\mathfrak{U})) & \xrightarrow{\delta} & \varinjlim \check{H}^1(\mathfrak{U},\mathscr{F}) & \longrightarrow & \varinjlim \check{H}^1(\mathfrak{U},\mathscr{G}) = 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow \\
0 \longrightarrow & \Gamma(X,\mathscr{F}) & \longrightarrow & \Gamma(X,\mathscr{G}) & \longrightarrow & \Gamma(X,\mathscr{R}) & \xrightarrow{\delta} & H^1(X,\mathscr{F}) & \longrightarrow & H^1(X,\mathscr{G}) = 0
\end{array}$$