Math 256B. Some Lemmas on Curves, and the End of the Proof of (II, 6.8)

This handout gives a few more lemmas on (nonsingular) curves, leading up to a more rigorous proof that the map of (II, 6.8) is finite if it is dominant.

Throughout this note, k is an algebraically closed field. Also, $t(C_K)$ is as in (II, 6.7) (i.e., a nonsingular projective curve over k whose function field is isomorphic to a given field K, finitely generated and of transcendence degree 1 over k).

Lemma 1. *Let A be an entire k-algebra of finite type such that $\operatorname{Spec} A$ is a nonsingular curve over k. Let $K = \operatorname{Frac} A$, and let $X = t(C_K)$. Then there is an open embedding $i\colon \operatorname{Spec} A \hookrightarrow X$ that induces the isomorphism $\operatorname{Frac} A \to K(X)$.*

Proof. Let X_0 be a projective closure of $\operatorname{Spec} A$ (I, Ex. 2.9). (In the language of schemes, this is done as follows. Choose a finite generating set $a_1, \dots, a_n \in A$ for A over k; then the associated ring surjection $k[x_1, \dots, x_n] \to A$ over k defined by $x_i \mapsto a_i$ for all i determines a closed embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}^n_k$. Then X_0 is the closure of the image of this map into $\mathbb{P}^n_k$, with reduced induced subscheme structure (where we identify $\mathbb{A}^n_k$ with the complement of the "hyperplane at infinity" H_0 as on p. 10 of Hartshorne).)

Moreover, letting $U_0 = \mathbb{P}^n_k \setminus H_0$, we have that $\operatorname{Spec} A$ is a closed subscheme of U_0 (via the usual identification of U_0 with $\mathbb{A}^n_k$), and the open subscheme $X_0 \cap U_0$ of X_0 is isomorphic to $\operatorname{Spec} A$, since both schemes are reduced and have the same (closed) image in U_0.

Now let $\pi\colon \widetilde{X} \to X$ be the normalization of X_0 (II, Ex. 3.8). Since $\operatorname{Spec} A$ is normal (it is nonsingular), π is an isomorphism over $X_0 \cap U_0$, so $\operatorname{Spec} A$ is also an open subscheme of $\widetilde{X}$. Also, since $\widetilde{X}$ is birational to $\operatorname{Spec} A$, we have

$$K(X) \cong K(\operatorname{Spec} A) = \operatorname{Frac} A .$$

Finally, since $\widetilde{X}$ is smooth and complete (it is finite, hence proper, over X_0), we have $\widetilde{X} \cong t(C_K)$, so one obtains the desired open embedding $\operatorname{Spec} A \hookrightarrow t(C_K) = X$. □

Lemma 2. *Let X be a variety over k, let $U = \operatorname{Spec} A$ be a nonempty open affine subset of X, and let $P \in X$ be a point not in U. Then there is a function $f \in A$ that does not extend to a regular function at P (i.e., $f \notin \mathscr{O}_{X,P}$).*

Proof. This is Question 3a on Homework 10. □

This next lemma is central to the main result of this handout.

Lemma 3. *Let X be a complete nonsingular curve over k, let Y be any curve over k, and let $f\colon X \to Y$ be a morphism with $f(X) = Y$. Let $V = \operatorname{Spec} B$ be any nonempty open affine subset of Y, and let A be the integral closure of B in $K(X)$. Then $\operatorname{Spec} A$ is isomorphic to an open subset U of X, compatible with the inclusion $A \subseteq K(X)$, and $U = f^{-1}(V)$.*

Proof. Clearly A is integrally closed with fraction field $K(X)$. Then $\dim \operatorname{Spec} A = 1$ and $\operatorname{Spec} A$ is normal, hence nonsingular. Also A is finite over B, hence of finite type over k. Thus $\operatorname{Spec} A$ is a nonsingular curve over k. The existence of U is then immediate from Lemma 1.

Next, since $X \cong t(C_{K(X)})$, the inclusion $B \subseteq A$ is compatible with the inclusion $K(Y) \subseteq K(X)$ induced by f. Therefore the natural map $\operatorname{Spec} A \to \operatorname{Spec} B$ coincides with $f\big|_U$. In particular, $f(U) \subseteq V$. In other words, $U \subseteq f^{-1}(V)$.

Therefore, if $U \ne f^{-1}(V)$, then there is a point $x \in X \setminus U$ such that $f(x) \in V$. Assume that x is such a point. Then, by Lemma 2, there is an $a \in A$ such that $a \notin \mathscr{O}_{X,x}$. Since a is integral over B, there are elements $b_0, \dots, b_{n-1} \in B$ such that

$$a^n + b_{n-1}a^{n-1} + \dots + b_0 = 0 .$$

But since $f(x) \in \operatorname{Spec} B$, the local ring $\mathscr{O}_{Y,f(x)}$ contains B, so $b_i \in \mathscr{O}_{Y,f(x)} \subseteq \mathscr{O}_{X,x}$ for all i. Thus a is integral over $\mathscr{O}_{X,x}$.

However, since X is nonsingular, it is normal, so $\mathscr{O}_{X,x}$ is integrally closed. In particular, $a \in \mathscr{O}_{X,x}$, a contradiction. Hence $U = f^{-1}(V)$. □

Finally, we get to the main result of this handout.

Proposition 4. *Let X be a complete nonsingular curve over k, let Y be any curve over k, and let $f\colon X \to Y$ be a morphism with $f(X) = Y$. Then f is a finite morphism.*

Proof. Let $V = \operatorname{Spec} B$ be any nonempty open affine subset of Y. By Lemma 3, $f^{-1}(V) = \operatorname{Spec} A$, where A is the integral closure of B in $K(X)$. By (I, 3.9A), A is finite as a module over B, and we are done. □