Math 256B. Homework 8

Due Wednesday 20 March

1(NC). Let $\mathscr{L}$ be a line sheaf on a scheme X. Suppose that there exist global sections $s \in \Gamma(X, \mathscr{L})$ and $t \in \Gamma(X, \mathscr{L}^\vee)$ such that $s \otimes t$ maps to 1 under the canonical isomorphism $\mathscr{L} \otimes \mathscr{L}^\vee \xrightarrow{\sim} \mathscr{O}_X$. Show that there exist isomorphisms $\phi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{O}_X$ and $\psi\colon \mathscr{L}^\vee \xrightarrow{\sim} \mathscr{O}_X$ such that $\phi(s) = \psi(t) = 1$.

2. Let $\mathscr{L}$ be an ample line sheaf on a projective variety X. Let $V_1, \dots, V_n$ be closed subvarieties, with $V_i \nsubseteq V_j$ for all $i \ne j$. Then there exists a positive integer m and $t_1, \dots, t_n \in \Gamma(X, \mathscr{L}^{\otimes m})$ such that $t_i\big|_{V_j} \ne 0$ for all $i \ne j$, and $t_i\big|_{V_i} = 0$ for all i.

   Here if X is a scheme, if $\mathscr{M}$ is a quasi-coherent sheaf on X, if Y is a closed subscheme of X with corresponding closed immersion $i\colon Y \to X$, and if s is a global section of $\mathscr{M}$, then $s\big|_Y$ is the global section of $i^*\mathscr{M}$ defined locally by $s \otimes 1$ (via the isomorphism of (II, Prop. 5.2e)). Or, tensoring the natural map $\mathscr{O}_X \to \mathscr{O}_Y$ with $\mathscr{M}$ gives a map $\mathscr{M} \to \mathscr{M} \otimes \mathscr{O}_Y$, and $s\big|_Y$ is the image of s under this map (note that $\mathscr{M} \otimes \mathscr{O}_Y \cong i^*\mathscr{M}$). It is also true (and you may use without proof) that if Y is integral, and if η is its generic point, then $s\big|_Y = 0$ if and only if $s_\eta \in \mathfrak{m}_\eta \mathscr{M}_\eta$, where $\mathfrak{m}$ is the maximal ideal in the local ring $\mathscr{O}_{X,\eta}$.

   [**Hint:** Think of the prime avoidance lemma in commutative algebra.]

3. Let A_0 be the subring of $k[x, y]$ generated by the set of all homogeneous polynomials of degree $\ne 1$, let $A = (A_0)_{x^2-1}$, and let $X = \operatorname{Spec} A$.
   (a). Show that X is separated, noetherian, integral, and regular in codimension one.
   (b). Show that X is not normal.
   (c). Show that the divisor $(x-1)$ equals zero as a Weil divisor, but that $x-1$ is not a regular function on X. (Note that $x-1$ is an element of $K(X) = k(x, y)$.)